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THEORY OF HIGH-ASPECT-RATIO PLANING SURFACES

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ABSTRACT

A high-aspect-ratio planing surface gliding on a stream of an infinitely deep, incompressible, inviscid and gravity-free fluid is treated. This complicated problem is decomposed into two relatively simpler boundary-value problems.

The near-field boundary-value problem is valid only in the neighborhood of the planing surface. The problem is solved by the classic hodograph method. The second-order inner problem is also shown to be a plane, irrotational flow and the solution is obtained by following the same procedure as given in the first-order inner solution.

The far-field boundary-value problem is valid only far away from the planing surface. The first-order outer solution is shown to be a trivial uniform flow. The outer velocity potential is defined in the whole space by harmonic continuation. The second-order solution is then shown to be similar to a lifting line solution.

The unknown strength of singularities is obtained by matching of the velocity potential. Then, a matching of the free-surface deflection provides a height reference for the planing surface. The location of the planing surface with respect to the undisturbed free surface is uniquely defined.

In order to obtain a unique second-order solution, it is necessary to solve the third-order outer solution. The detail of this solution is presented.

A numerical solution for a planing plate of arbitrary angle of attack is presented. A downwash correction is also included.

It is shown mathematically that the present theory can be applied to V-shape or general shape planing surfaces with curvature in the spanwise direction.

CONTENTS

CONTENTS (CONT'D)

SYMBOLS

Primed variables are dimensional; unprimed variables are dimensionless

A_0, A_1	constants in inversion formula
AR	aspect-ratio
a_0, a_1	jet thickness (outer variables)
a^*	$=a_0+a_1$
b_0, b_1	jet in ζ-plane
b^*	$=b_0+b_1$
c_0, c_1	stagnation point in ζ-plane
c^*	$=c_0+c_1$
F_0, F_1	complex potential
H	height of the trailing edge above the undisturbed free surface (inner)
h	height of the trailing edge above the undisturbed free surface (outer)
L	local chord length (inner)
ℓ	local chord length (outer)
M_0, M_1	constants
N	unit vector in a plane z=constant, normal to body contour in that cross-section
N_0, N_1	free-surface elevation (inner)
p	pressure
p_∞	atmospheric pressure
R_0, R_1	constants
S_f	free-surface

SYMBOLS (CONT'D)

U	constant uniform velocity
W_0, W_1	complex velocity
x, y, z	rectangular coordinates (outer)
X, Y, Z	rectangular coordinates (inner)
Z	complex variable
α	angle of attack
β	jet angle
γ, μ, λ	strength of singularities
δ_0, δ_1	jet thickness (inner)
ε	small parameter
ζ	complex variable $\zeta=\xi+i\eta$
η	free-surface elevation (outer)
ρ	mass density
τ_0, τ_1	lift force per-unit-span
Φ	velocity potential (inner)
ϕ	velocity potential (outer)
Ψ	stream function
i	imaginary unit
Re	real part
∇^2	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

I. INTRODUCTION

I. 1. Introduction and Review

With the appearance of seaplanes and high speed planing boats, the problems associated with planing surfaces became practical problems. It is interesting to note that the first experiment with a flat planing surface was carried out in 1912,[1] which was followed by a series of both theoretical and experimental investigations.

At rest and low forward speeds the planing boat is supported by buoyancy force. As speed increases, the center of gravity gradually begins to rise. The center of pressure on the bottom shifts toward the nose and increases the trim by the stern. With further increase in the speed, the wetted area decreases, the mean draft of the hull becomes so slight that most of its weight is supported by the dynamic lift acting on its bottom, and the hydrostatic force is very small.[2] This condition is called *planing* and the bottom surface which is in contact with water is called the *planing surface*.

Wagner,[3] in his fundamental paper, showed that for infinitely small angles of attack, with irrotational and gravity-free motion, the flow on the lower surface of a planing surface is comparable with that on the lower surface of an airfoil. The lift of a planing surface is approximately half as great as that of an infinitely thin airfoil whose plan corresponds to the wetted surface. Sottorf[4,5] and Sambraus[6]

indicated a satisfactory agreement for moderately small angles of attack between their test data on flat planing surfaces and Wagner's theory. However, Wagner's method is approximate and, strictly, only applies when the angle of incidence of the gliding plate to the stream is moderately small; moreover, his solution requires that the direction of the jet or spray which is formed should be parallel to the length of the plate. In the middle 1930's, A. E. Green[7,8] presented several papers on this subject. He treated a two dimensional plate gliding on the surface of a gravity-free stream of infinite depth. He was able to obtain a complete solution of a two dimensional gravity-free planing-plate problem, satisfying the non-linear free-surface boundary conditions exactly.

However, his solution is not unique.[9] There is another anomaly in Wagner's and Green's theories of infinitely deep water: their theories imply that the free-surface elevation infinitely far upstream and downstream is depressed an infinite amount.

The principal difficulty of a free surface problem is that both the dynamic and kinematic free-surface boundary conditions are non-linear. Worse than that, the free surface on which the conditions are to be applied is not known *a priori*; it has to be found as part of the problem solution. To overcome this difficulty, another approach is found in the literature. Assume the disturbance is

small; then the problem can be linearized. It is assumed that the error is of high order if the free surface boundary conditions and the body condition are applied on the undisturbed free surface instead of on actual positions of free surface and body surface. Then the higher order terms in the free surface boundary conditions are neglected compared to the linear term. In this way the two dimensional boundary-value problem is reduced to an integral equation which was first solved by Sretensky[10], later by Maruo[11] and Cumberbatch.[12] Lamb[13] considered the two dimensional flow due to the application of a pressure distribution on the surface of a stream. He computed two simple pressure distributions for which the integrals can be evaluated to give the shape of the planing surface. Such linearized solutions give a good approximation to the expected physical behavior, except in the neighborhood of the plate, where the flow near the stagnation point and the spray sheet can not be regarded as small perturbations of the uniform stream.

Rispin[14] and Wu[15] used the singular perturbation method to solve the two dimensional non-linear planing problem by including gravity effects. They obtained a complete solution which includes an inner expansion and an outer expansion. Both anomalies in Wagner's and Green's classic solutions were removed.

In the experimental approach, Shoemaker[16] made a test

for V-shaped as well as flat-plate planing surfaces. Korvin-Kroukovsky[17] gave an empirical formula for lift calculation. Shuford[18,19] gave a review of planing theory and experiment of rectangular flat plates. Using the semi-empirical formula, the effects of cross section and plan form were included.

Progress in developing a three dimensional theoretical approach is rather slow. Using linearized theory, Maruo[20] obtained an approximate solution from the integral equation for high-and low-aspect-ratio planing-surface problems. A low-aspect-ratio planing surface has been treated as a slender body by Tulin,[21] without matched expansions and by Ogilvie,[22] with matched expansions.

A low-aspect-ratio planing surface is relatively inefficient in producing lift force; more area has to be provided for a given lift, with a resulting penalty in lift-drag ratio compared to that of a high-aspect-ratio surface such as a hydrofoil. Clement[23] proposed a new plan form with a re-entrant step in his planing boat design. Brown and Van Dyck[24] made an experimental investigation on this high-aspect-ratio re-entrant step planing surface, with an encouraging result. Current development of practical stabilizers[25] is proceeding rapidly enough that within a few years a high-aspect-ratio planing boat will become practical.

In his experimental studies on a rectangular-plan-form, high-aspect-ratio, planing plate, Mottard[26] found that an

instability quite similar to the flutter of an airfoil takes place even if only a single degree of freedom, such as heave motion, is present. This is a rather interesting result. In aerodynamics, the occurrence of flutter normally requires that two vibrational degrees of freedom be involved. Ogilvie[27] linearized the lifting-surface problem in the manner of Wagner and solved this two dimensional hydrodynamic problem in the presence of heave motion. He obtained a parameter ν, a reduced frequency, which characterized the occurrence of instability. His theoretical result agrees very well with Mottard's experimental data. In order to solve a three dimensional instability problem, a three dimensional steady state hydrodynamic problem has to be solved first. This is the aim of the present work.

I. 2. Scope and Nature of the Solution

The trailing edge at the step of a planing boat is a straight line. The contours of the bottom lines of the planing surface in the immediate neighborhood of the step closely approximate those of a flat plate. As the speed of the planing surface increases, the wetted-aspect-ratio, AR, increases rapidly. The hydrostatic force and friction force become negligible compared to the increased hydrodynamic force. In the present study, only the limiting high-speed case of inviscid flow will be considered, that is, the effects of gravity and viscosity are neglected.

In order to apply the hodograph method, a planing surface without camber in the longitudinal direction is considered.

Let the aspect-ratio AR be defined as

$$AR = \frac{\text{Span}^2}{\text{Wetted Planing Area}} = \frac{\text{Span}}{\text{Mean Wetted Chord Length}} .$$

For a high-aspect ratio planing surface, it is implied that the maximum dimension in the z direction is much greater than the maximum dimension in the cross-sections. Then a small parameter ε can be defined as the inverse of high-aspect ratio:

$$\varepsilon = \frac{1}{AR} = \frac{\text{Mean Wetted Chord Length}}{\text{Span}}$$

As aspect-ratio AR becomes larger and larger, the span grows bigger and bigger *if the mean wetted length is fixed.* At its limit as AR approaches infinity, the span approaches infinity. For an infinite span the flow will be two dimensional. This is exactly the lowest-order near-field description. The characteristic length will be the chord length at each section. On the other hand if ε becomes smaller and smaller, the scale of mean wetted length becomes smaller and smaller *if the span is fixed.* At its limit as ε approaches zero, the planing surface shrinks to a line and uniform flow is recovered. This is exactly the lowest-order far-field description. The characteristic

length will be the span. There are two different reference length scales working at the same time. It suggests that a singular perturbation method can be applied to solve this problem.

In the present work, the method of matched asymptotic expansions is applied to solve a high-aspect-ratio planing surface problem. The full boundary-value problem is decomposed into near-field and far-field boundary-value problems. The first-order near-field problem with respect to the small parameter ε is found to be a plane irrotational-flow problem. The solution is obtained by the hodograph method. The first-order outer solution represents a uniform stream. The second-order inner problem is shown to be a plane irrotational-flow problem again. The second-order outer problem is shown to be similar to the lifting line problem. Following the matching principle[28] as outlined by Van Dyke, unknown parameters in the near- and far-field problem are determined. Finally, a unique solution is obtained through matching of the free-surface deformation. The depth of submergence can thus be properly defined. The anomaly in the Wagner's and Green's gravity-free infinite depth solution is thus removed by the present work. Three dimensional effects are shown to provide a height reference. Thus present theory is applicable for heavy loading as well as light loading on the planing surface.

II. FORMULATION OF THE PROBLEM

II. 1. Statement of the Problem

The problem treated here is a three dimensional flow generated by a high-aspect-ratio planing surface gliding with a constant speed U' at an arbitrary angle on the free surface of an infinitely deep weightless fluid. The fluid is assumed to be incompressible and inviscid and the flow irrotational. Instead of a planing surface moving on calm water, a uniform stream of speed U' at infinity in the direction of positive x-axis is superimposed. Then a steady motion is obtained. Take the coordinate system with the origin on the undisturbed free surface and choose the x-axis in the direction of the uniform stream. The positive y-axis is taken vertically upwards and the z-axis is perpendicular to the xy-plane as shown in Figure 2.1.

Let the span be b' and the planing surface located in the segment $|z| \leq b'/2$. Let the chord length of the planing surface be $\ell'(z')$. The function $\ell'(z')$ must be continuous and have a continuous derivative, at the tip as elsewhere; otherwise singularities will occur and the region near the tips will need a special treatment.[28] Let the angle of incidence be α, the aspect ratio of the planing surface be AR, and the small parameter be ε, which is the inverse of the aspect ratio. Let the vertical

distance between the trailing edge and the undisturbed free-surface level be $h'(z')$. To simplify the analysis, it is assumed here that the trailing edge is a straight horizontal line. This assumption implies that h' is independent of z. Therefore the uncambered planing surface becomes a planing plate. In Chapter 8, this restriction will be relaxed so that the bottom of the planing surface can be V-shaped or even curved in the spanwise direction.

Assume that the angle of incidence α, the depth of submergence h' and the shape of the chord length distribution $\ell'(z')$ are given. The solution of the hydrodynamic problem of the planing surface is to be determined.

II. 2. Boundary Value Problem for the Planing Plate

The fluid is assumed to be ideal and the flow irrotational. There exists a velocity potential $\phi'(x',y',z')$ which satisfies the Laplace equation

$$\text{(L)} \qquad \nabla^2\phi' = 0 \qquad \text{in the fluid region}$$

Assume that the free surface deflection can be specified by the equation

$$y' = \eta'(x',z') \qquad \text{on the free surface } S_f$$

The function η' is unknown; it is a part of the problem to be solved.

There are two conditions to be satisfied on the free surface. Let the velocity be the positive gradient of the velocity potential, $\bar{q}' = \nabla\phi'$, where $\bar{q}'$ is the velocity vector. The first condition is called the dynamic boundary condition. It states that the pressure is constant on the free surface, equal to the atmospheric pressure:

$$p'(x',\eta'(x',z'),z') = p_\infty' \qquad \text{on } S_f$$

The Bernoulli equation gives

$$\frac{p'}{\rho'} + \frac{1}{2}(\phi_{x'}'^2 + \phi_{y'}'^2 + \phi_{z'}'^2) = \frac{p_\infty'}{\rho'} + \frac{1}{2}U'^2 \quad ,$$

where p_∞' is the atmospheric pressure, p' is the local pressure in the fluid, ρ' is the density of the fluid. An assumption is made that no disturbance exists far upstream at infinity. On the free surface, the Bernoulli equation becomes:

$$\text{(F1)} \qquad \phi_{x'}'^2 + \phi_{y'}'^2 + \phi_{z'}'^2 = U'^2 \qquad \text{on } S_f$$

The second condition on the free surface is called

the kinematic boundary condition. It states that a particle once in the free surface remains in the free surface. Mathematically, it can be stated:

$$\frac{D}{Dt}(S_f) = \frac{D}{Dt}[\eta'(x',z') - y'] = 0 \qquad \text{on } S_f ,$$

where $\frac{D}{Dt}$ denotes the substantial derivative. The above equation gives:

$$\text{(F2)} \qquad \phi'_{x'}\eta'_{x'} - \phi'_{y'} + \phi'_{z'}\eta'_{z'} = 0 \qquad \text{on } S_f,$$

Equations (F1) and (F2) are two non-linear conditions to be satisfied on an unknown boundary S_f.

Next, a body boundary condition must be satisfied. A body surface might be defined by the equation

$$\overline{s}' = s'(x',z') + y' = 0$$

The kinematic boundary condition on the body would be

$$\phi'_{x'}\overline{s}'_{x'} - \phi'_{y'} + \phi'_{z'}\overline{s}'_{z'} = 0 \qquad \text{on } y' = - s'(x',z').$$

For the planing plate, for which:

$$\bar{s}' = x' \tan \alpha + y' = 0,$$

the body condition is :

$$\text{(B)} \quad \phi'_{x'} \sin \alpha + \phi'_{y'} \cos \alpha = 0 \qquad \text{on the plate}$$

In airfoil theory, a Kutta condition is applied to give a finite velocity at the trailing edge. In the planing problem, this condition requires that the flow separates smoothly from the the trailing edge of the planing surface.

Finally, there is a radiation condition to be satisfied to guarantee a unique solution. This condition states that there is no disturbance far upstream.

The planing plate problem is thus mathematically formulated. The governing equation is the Laplace equation subject to two free-surface boundary conditions, a body condition and a radiation condition.

It is more convenient to work on the problem by making the variables and the equations dimensionless.

Let the span be normalized to be of length 2 and located in the segment $|z| \leq 1$. And define the following variables:

$$b = \frac{b'}{b'/2} = 2$$

$$x,y,z = \frac{x'}{b'/2}, \; \frac{y'}{b'/2}, \; \frac{z'}{b'/2}$$

$$q = \frac{q'}{U'} \quad (u,v,w = \frac{u'}{U'}, \frac{v'}{U'}, \frac{w'}{U'})$$

$$\eta = \frac{\eta'}{b'/2}, \quad h = \frac{h'}{b'/2}, \quad \ell = \frac{\ell'}{b'/2}$$

$$p = \frac{p'}{\rho' U'^2}, \quad \phi = \frac{\phi'}{U' b'/2}$$

The full boundary-value problem in dimensionless form becomes:

$$\text{(L)} \quad \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{in the fluid region} \quad (2.1)$$

$$\text{(F1)} \quad \phi_x^2 + \phi_y^2 + \phi_z^2 = 1 \quad \text{on } S_f \quad (2.2)$$

$$\text{(F2)} \quad \phi_x \eta_x - \phi_y + \phi_z \eta_z = 0 \quad \text{on } S_f \quad (2.3)$$

$$\text{(B)} \quad \phi_x \sin\alpha + \phi_y \cos\alpha = 0 \quad \text{on plate} \quad (2.4)$$

$$\text{(R)} \quad \phi_x = 1 \; ; \; \phi_y = \phi_z = 0 \quad \text{as } x \to -\infty,\ y \to -\infty \quad (2.5)$$

Now, consider the flow field far away from the planing plate. There is an incident flow which, at infinity, is uniform in the x-direction. In the absence of the body, the velocity potential is x. As $\varepsilon \to 0$, the body shrinks to a line normal to the incident flow, so the first term in the far

field description represents simply a uniform stream, upon which a line of higher order singularities is superimposed. On the other hand, consider the flow field in the neighborhood of the planing plate. The flow in the whole vicinity of the planing surface can not be regarded as a small perturbation of the uniform stream for arbitrary angle α

As aspect-ratio AR approaches infinity, the flow in the near field acts as if ℓ is fixed and the scale of span b approaches infinity at each section. In this sense, it is similar to the flow of infinite span and the flow field is two dimensional to the lowest order. So the singular perturbation method can be applied to reduce the original complicated free-surface problem into two relatively simpler problems. The outer solution is valid only far away from the planing surface, while the inner solution is valid only near the planing surface. An approximate solution, valid everywhere, in the form of a power series* in ε, will be obtained by calculating successive solutions of the outer and inner problems and matching them according to the matching principle:[29]

m-term inner expansion of (n-term outer expansion)

= n-term outer expansion of (m-term inner expansion). (2.6)

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* The power series contains "logarithmic" terms. However, $\log \varepsilon$ will be treated as O(1) as suggested by Van Dyke.[29] Otherwise the Van Dyke rule leads to contradictory results.

Here m and n are any two integers. This matching is necessary in order to obtain inner boundary conditions for the outer solution and vice versa.

II. 3. The Far Field Boundary-Value Problem

Let x, y, z be the outer variables which correspond to the natural variables. Let ϕ be the velocity potential for the outer problem. The xz-plane is located on the undisturbed free surface with origin located at the intersection of the planing plate and the undisturbed free surface, as shown in Figure 2.2. The outer problem corresponds to the flow field far away from the plate. There the detail of the planing plate and the Kutta condition become meaningless, while the Laplace equation, dynamic and kinematic free-surface conditions, and radiation condition must still be satisfied. The boundary value problem for the far field can thus be formulated as following:

(L) $$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{in fluid} \qquad (2.7)$$

(F1) $$\phi_x^{\,2} + \phi_y^{\,2} + \phi_z^{\,2} = 1 \quad \text{on } S_f \qquad (2.8)$$

(F2) $$\phi_x \eta_x - \phi_y + \phi_z \eta_z = 0 \quad \text{on } S_f \qquad (2.9)$$

(R) $$\phi_x = 1 \;;\; \phi_y = \phi_z = 0 \quad \text{as } x \to -\infty,\ y \to -\infty \qquad (2.10)$$

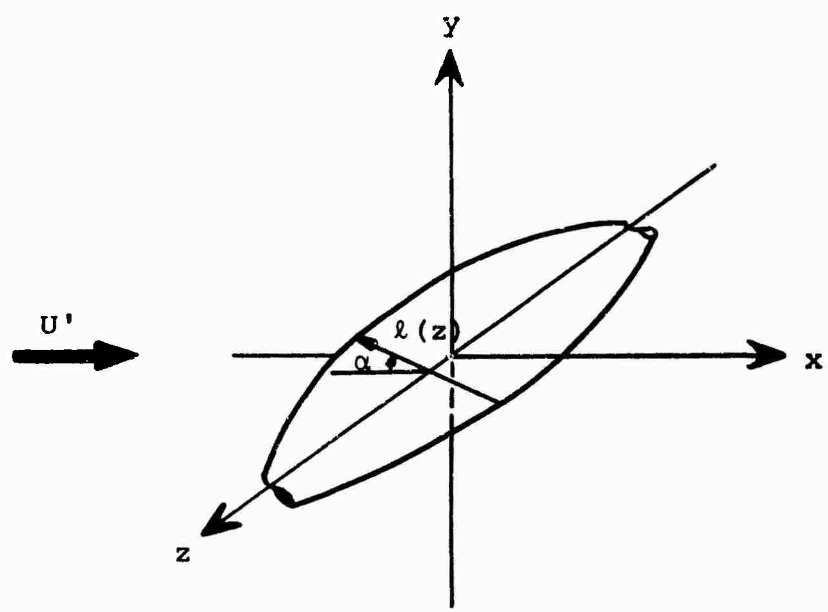

Figure 2.1 Chord Length Distribution

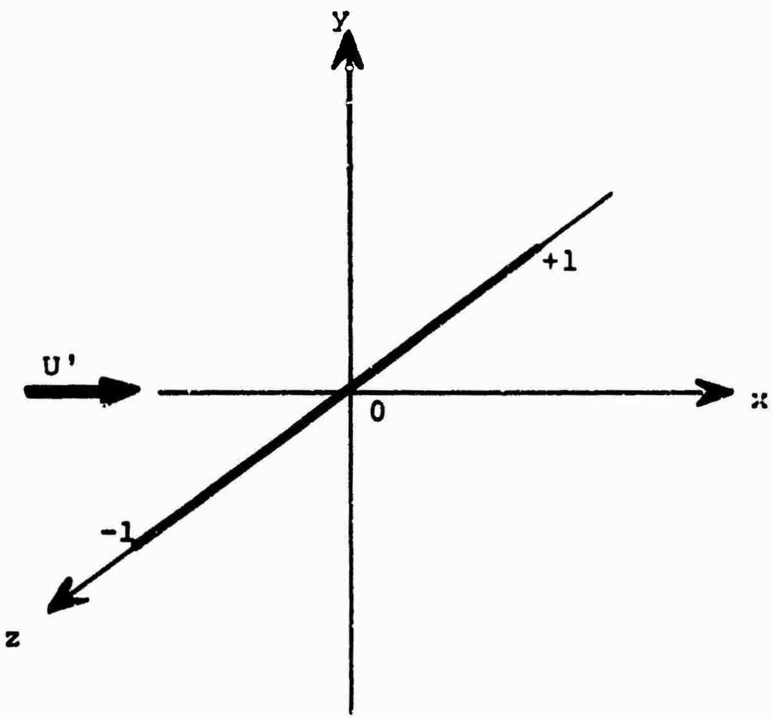

Figure 2.2 Outer Limit

Next, assume that there exists an asymptotic expansion valid in the far field:

$$\phi(x,y,z) \sim \sum_{n=0}^{N} \phi_n(x,y,z;\varepsilon) \qquad (2.11)$$

where $\phi_{n+1} = o(\phi_n)$

as $\varepsilon \to 0$ for fixed x,y,z

Also assume the existence of a complementary asymptotic expansion for the free-surface elevation

$$\eta \sim \sum_{n=0}^{N} \eta_n(x,z;\varepsilon) \qquad (2.12)$$

where $\eta_{n+1} = o(\eta_n)$

as $\varepsilon \to 0$ for fixed x,z

Substitute equations (2.11) and (2.12) into equations (2.7) - (2.10) to give:

(L) $$\phi_{0xx} + \phi_{0yy} + \phi_{0zz} + \phi_{1xx} + \phi_{1yy} + \phi_{1zz} + \phi_{2xx} + \phi_{2yy} + \phi_{2zz} + \ldots . = 0 \quad (2.13)$$

(F1) $$\phi_{0x}^{2} + \phi_{0y}^{2} + \phi_{0z}^{2} + 2\phi_{0x}\phi_{1x} + 2\phi_{0y}\phi_{1y} + 2\phi_{0z}\phi_{1z} + \phi_{1x}^{2} + 2\phi_{0x}\phi_{2x} + \phi_{1y}^{2} + 2\phi_{0y}\phi_{2y} + \phi_{1z}^{2} + 2\phi_{0z}\phi_{1z} + 2\phi_{1x}\phi_{2x} + 2\phi_{1y}\phi_{2y} + 2\phi_{1z}\phi_{2z} + \ldots\ldots = 1 \quad \text{on } S_f \qquad (2.14)$$

$$\text{(F2)}\quad \phi_{0x}\eta_{0x} - \phi_{0y} + \phi_{0z}\eta_{0z} + \phi_{0x}\eta_{1x} + \phi_{1x}\eta_{0x} - \phi_{1y}$$

$$+ \phi_{0z}\eta_{1z} + \phi_{1z}\eta_{0z} + \phi_{1x}\eta_{1x} + \phi_{2x}\eta_{0x} + \phi_{0x}\eta_{2x}$$

$$- \phi_{2y} + \phi_{0z}\eta_{2z} + \phi_{1z}\eta_{1z} + \phi_{2z}\eta_{0z} + \ldots = 0$$

$$\text{on } S_f \qquad (2.15)$$

$$\text{(R)}\quad \begin{aligned} \phi_{0x} + \phi_{1x} + \phi_{2x} + \ldots\ldots &= 1 \\ \phi_{0y} + \phi_{1y} + \phi_{2y} + \ldots\ldots &= 0 \quad \text{as } x\to-\infty,\ y\to-\infty \qquad (2.16) \\ \phi_{0z} + \phi_{1z} + \phi_{2z} + \ldots\ldots &= 0 \end{aligned}$$

Equations (2.13) - (2.16) represent the mathematical expression for the outer boundary-value problem.

II. 4. The Near Field Boundary-Value Problem

In the neighborhood of a high-aspect ratio planing plate, the flow acts as if it is nearly two dimensional. In order to investigate details of this flow field, one thus shall have to become more and more nearsighted as the limit is approached. Mathematically the x-and y-coordinates have to be magnified in order to have flow variables change on a reasonable scale. Let x, y, z

be the natural variables.

Let X, Y, Z be the magnified inner variables with origin at the trailing edge, Φ the inner velocity potential. Then

$$X = \frac{x + h \cot \alpha}{\varepsilon}, \qquad Y = \frac{y - h}{\varepsilon}, \qquad Z = z \qquad (2.17a)$$

as shown in Figure 2.3

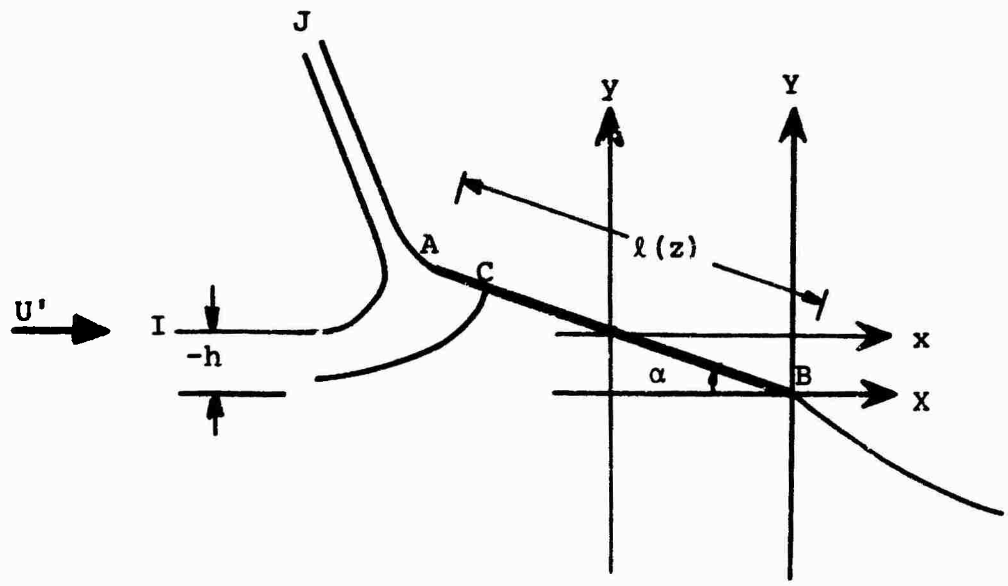

Figure 2.3 Inner Limit

Also define

$$H = \frac{h}{\varepsilon}, \qquad N = \frac{\eta}{\varepsilon}, \qquad L = \frac{\ell}{\varepsilon} \qquad (2.17b)$$

A physical quantity, such as velocity, should not be changed

by magnified variables:

$$\phi_x = O(\Phi_X)$$

The inner problem is valid only in the neighborhood of the planing plate. The radiation condition is lost. However, the Laplace equation, dynamic and kinematic free-surface conditions, body boundary condition and the Kutta condition must still be true. The boundary-value problem for the near field can thus be formulated as follows:

$$\text{(L)} \quad \Phi_{XX} + \Phi_{YY} + \varepsilon^2\Phi_{ZZ} = 0 \quad \text{in fluid} \qquad (2.18)$$

$$\text{(F1)} \quad \Phi_X^2 + \Phi_Y^2 + \varepsilon^2\Phi_Z^2 = 1 \quad \text{on } S_f \qquad (2.19)$$

$$\text{(F2)} \quad \frac{1}{\varepsilon}\Phi_X N_X - \Phi_Y + \varepsilon\Phi_Z N_Z = 0 \quad \text{on } S_f \qquad (2.20)$$

$$\text{(B)} \quad \Phi_X \sin\alpha + \Phi_Y \cos\alpha = 0 \quad \text{on body} \qquad (2.21)$$

(K) The flow separates smoothly from the trailing edge

Next assume the existence of the following asymptotic series with respect to ε

$$\Phi \sim \sum_{n=0}^{N} \Phi_n(X,Y,Z;\varepsilon)$$

$$\text{where} \quad \Phi_{n+1} = o(\Phi_n) \qquad (2.22a)$$

$$\text{as} \quad \varepsilon \to 0 \text{ for fixed } X,\ Y,\ Z$$

$$N \sim \sum_{n=0}^{N} N_n(X,Z;\varepsilon)$$

$$\text{where} \quad N_{n+1} = o(N_n) \qquad (2.22b)$$

$$\text{as} \quad \varepsilon \to 0 \text{ for fixed } X,\ Z$$

Substitute equations (2.22a) and (2.22b) into equations (2.18) - (2.21) to give:

(L)
$$\Phi_{0XX} + \Phi_{1XX} + \Phi_{2XX} + \Phi_{3XX} + \cdots\cdots$$
$$+\Phi_{0YY} + \Phi_{1YY} + \Phi_{2YY} + \Phi_{3YY} + \cdots\cdot \qquad (2.23)$$
$$+ \varepsilon^2(\Phi_{0ZZ} + \Phi_{1ZZ} + \Phi_{2ZZ} + \cdots\cdot) = 0 \text{ in fluid}$$

(F1)
$$\Phi_{0X}^2 + 2\Phi_{0X}\Phi_{1X} + 2\Phi_{0X}\Phi_{2X} + \Phi_{1X}^2 + \cdots\cdot$$
$$+ \Phi_{0Y}^2 + 2\Phi_{0Y}\Phi_{1Y} + \Phi_{1Y}^2 + 2\Phi_{0Y}\Phi_{2Y} + \cdots\cdot$$
$$+\varepsilon^2(\Phi_{0Z}^2 + 2\Phi_{0Z}\Phi_{1Z} + \Phi_{1Z}^2 + 2\Phi_{0Z}\Phi_{2Z} + \cdots\cdot) = 1$$
$$\text{on } S_f \qquad (2.24)$$

(F2)
$$\frac{1}{\varepsilon}(\Phi_{0X}N_{0X} + \Phi_{0X}N_{1X} + \Phi_{1X}N_{0X} + \Phi_{1X}N_{1X}$$
$$+ \Phi_{0X}N_{2X} + \cdots\cdot) - (\Phi_{0Y} + \Phi_{1Y} + \Phi_{2Y} + \Phi_{3Y} + \cdots\cdot)$$
$$+ \varepsilon(\Phi_{0Z}N_{0Z} + \Phi_{0Z}N_{1Z} + \Phi_{1Z}N_{0Z} + \Phi_{1Z}N_{1Z}$$

$$+ \Phi_{0Z}N_{2Z} + \ldots.) = 0 \qquad (2.25)$$

(B) $$(\Phi_{0X} + \Phi_{1X} + \Phi_{2X} + \ldots.)\sin\alpha + (\Phi_{0Y} + \Phi_{1Y} + \Phi_{2Y} + \ldots.)\cos\alpha = 0 \quad \text{on body} \qquad (2.26)$$

This is the non-linear near-field surface problem.

Now, consider the lowest order of this inner problem. According to the definition of asymptotic expansion, Φ_{1XX} is of higher order than Φ_{0XX} in equation (2.23) and can be discarded. Therefore the possible lowest-order Laplace equation is:

$$\Phi_{0XX} + \Phi_{0YY} + \varepsilon^2\Phi_{0ZZ} = 0$$

However, according to the definition, Φ_{0ZZ} is of the same order as Φ_{0XX} and Φ_{0YY}. Finally, it gives:

(L) $$\Phi_{0XX} + \Phi_{0YY} = 0 \quad \text{in fluid} \qquad (2.27)$$

Similar argument leads to the following boundary conditions.

(F1) $$\Phi_{0X}{}^2 + \Phi_{0Y}{}^2 = 1 \quad \text{on } S_f \qquad (2.28)$$

(F2) $$\Phi_{0X}N_{0X} - \varepsilon\Phi_{0Y} = 0 \quad \text{on } S_f \qquad (2.29)$$

(B) $$\Phi_{0X} \sin\alpha + \Phi_{0Y} \cos\alpha = 0 \qquad \text{on body} \qquad (2.30)$$

This is the lowest-order inner boundary value problem, which will be called the *first-order inner problem* throughout the entire work.

III. FIRST-ORDER SOLUTION

III. 1. First-Order Inner Solution:

The inner boundary-value problem was mathematically formulated in the previous chapter and then was expanded in an asymptotic series with respect to ε. To the first order the boundary value problem is:

$$\text{(L)} \qquad \Phi_{0XX} + \Phi_{0YY} = 0 \qquad \text{in fluid} \qquad (3.1)$$

$$\text{(F1)} \qquad {\Phi_{0X}}^2 + {\Phi_{0Y}}^2 = 1 \qquad \text{on } S_f \qquad (3.2)$$

$$\text{(F2)} \qquad \Phi_{0X} N_{0X} - \varepsilon \Phi_{0Y} = 0 \qquad \text{on } S_f \qquad (3.3)$$

$$\text{(B)} \qquad \Phi_{0X} \sin\alpha + \Phi_{0Y} \cos\alpha = 0 \qquad \text{on body} \qquad (3.4)$$

(K) Kutta Condition

The governing differential equation is a two dimensional Laplace equation at each section. The boundary conditions are all two dimensional. Therefore, the first order inner problem concerns a plane irrotational flow. The dynamic boundary condition (F1) states that the magnitude of the velocity on the free-surface is constant

and known. The body condition (B) states that the direction of the flow on the body surface is known. This is a free streamline problem and can be solved by the classical hodograph method.

It is noted that Z coordinate appears only implicitly in this first-order inner problem. It will be shown in the next chapter that the Z coordinate also appears only implicitly in the second order inner problem. Therefore, in order to stay with the conventional symbol, Z will be defined as a complex variable $Z=X+iY$ through the whole analysis.

Experiments[30] indicate that near the planing surface the flow shown by tests presents a true jet or spray character; i.e., at some distance ahead of the planing body the water surface is practically undisturbed, while immediately forward of the planing body the water is thrown off in a spray. In the region behind the planing body, there is no spray separating at the trailing edge at large planing speeds; the water flows off smoothly from the trailing edge.

From this experimental observation, the first-order inner problem can be stated as following: An infinitely long flat plate is held at an arbitrary angle α on the free surface of an otherwise uniform stream of an infinitely deep weightless fluid. The plate is of length L. The

origin is located at the trailing edge B which is at a depth H from the undisturbed free surface. Y-axis is vertically upwards, X-axis is in the direction of uniform stream. The plate is $\overline{AB}$. The flow comes from upstream infinity and divides along the stagnation streamline which branches off at the stagnation point C. Above this streamline, the flow shoots off as a jet J. Below this streamline, the flow separates smoothly from the trailing edge B as shown in Figure 2.3. From equation (3.1) the velocity potential $\Phi_0(X,Y)$ is harmonic and its conjugate $\Psi_0(X,Y)$ is the stream function. Define the complex potential $F_0(Z)$:

$$F_0(Z) = \Phi_0(X,Y) + i\Psi_0(X,Y)$$

Let $W_0(Z)$ be the derivative of $F_0(Z)$ with respect to Z:

$$W_0(Z) = \frac{dF_0(Z)}{dZ} = \Phi_{0x} - i\Phi_{0Y} = U_0 - iV_0$$

where U_0 and V_0 are velocity components in the X and Y directions, respectively. $W_0(Z)$ is called the complex velocity. Both $F_0(Z)$ and $W_0(Z)$ are analytic functions.

The dynamic free-surface condition (F1) gives:

$$U_0^2 + V_0^2 = 1 \qquad \text{on } S_f \qquad (3.5)$$

The kinematic condition gives:

$$\frac{1}{\varepsilon}\frac{dN_0}{dX} = \frac{\Phi_{0Y}}{\Phi_{0X}} = \frac{V_0}{U_0} \qquad \text{on } S_f \qquad (3.6)$$

The body condition gives:

$$\frac{V_0}{U_0} = -\tan\alpha \qquad (3.7)$$

The first-order inner problem is thus expressed in natural variables. This problem will be transformed onto the lower half of the auxiliary ζ-plane as shown in Figure 3.2. The problem is then solved in the ζ-plane and finally the solution is transformed back to the physical plane by the inversion formula.

Let the potential function Φ_0 be zero at C and the dividing streamline be $\Psi_0=0$. Consider the F_0-plane shown in Figure 3.1. IJ is the upstream free surface. If a_0 is the breadth of the jet at a great distance expressed in the outer variables, then the breadth in inner variables is $\delta_0 = \frac{a_0}{\varepsilon}$. Along IJ, the stream function Ψ_0 is

$$\Psi_0 = \frac{a_0}{\varepsilon} = \delta_0$$

The polygon in the F_0-plane can now be transformed onto the lower half of ζ-plane. Let A, B correspond to $\zeta = -1, +1$ respectively. The chord length is represented by the segment $|\xi| < 1$ on the real axis $\eta = 0$, where $\zeta = \xi + i\eta$. Let J, C then be mapped onto $\zeta = -b_0, c_0$. The free surface is thus mapped onto the real axis too. The diagram of the ζ-plane is shown in Figure 3.1. The Schwarz-Christoffel transformation gives:

$$\frac{dF_0}{d\zeta} = K \frac{\zeta - c_0}{\zeta + b_0}$$

Integration gives:

$$F_0(\zeta) = \frac{\delta_0}{\pi(b_0+c_0)} [\zeta - c_0 - (b_0+c_0) \log\frac{\zeta+b_0}{b_0+c_0}] \qquad (3.8)$$

This is Green's solution.[7] (See Milne-Thomson[32])

Next, define the mapping function $H_0(\zeta)$

$$H_0(\zeta) = \frac{dF_0}{d\zeta} = \frac{\delta_0}{\pi(b_0+c_0)} \cdot \frac{\zeta-c_0}{\zeta+b_0} \qquad (3.9)$$

Then

$$W_0(Z(\zeta)) = \frac{dF_0}{dZ} = H_0(\zeta)\frac{d\zeta}{dZ} \qquad (3.10)$$

From equation (3.10), integration gives

$$Z = \int_{1}^{\zeta} \frac{H_0(\zeta')}{W_0(Z(\zeta'))} d\zeta' \tag{3.11}$$

The origin is located at the trailing edge to give the lower limit of the integral to be 1. The equation (3.11) relates the ζ-plane and the physical plane. The inner solution found in the ζ-plane will be transformed back to the physical plane by the equation (3.11), which is called the inversion formula.

The equations (3.5), (3.6), and (3.7) state that the flow is bounded by a straight rigid boundary (planing surface) and a curvilinear free streamline (free surface). The complete boundary can be transformed into a simple closed polygon by a Kirchhoff function.

Define the complex variable Q_0 to be

$$Q_0 = - \log(W_0) = \tau_0 + i\Theta_0 \tag{3.12a}$$

Let q be the speed in the flow field. Then

$$Q_0 = - \log q - i\arg W_0 \tag{3.12b}$$

Equations (3.12a) and (3.12b) give:

$$q^2 - e^{-2\tau_0} = 0$$

$$\Theta_0 + \arg W_0 = 0$$

On the free surface, q=1, which gives:

$$1 - e^{-2\tau_0} = 0 \qquad \text{on } S_f \qquad (3.13a)$$

On the body surface,

$$\Theta_0 = - \arg W_0 = -\alpha \qquad \text{on } \overline{CB} \qquad (3.13b)$$

$$\Theta_0 = \pi - \alpha \qquad \text{on } \overline{AC}$$

Next, express these boundary conditions on the real axis, $\eta=0$, of the ζ-plane. Equation (3.13) becomes:

$$1 - e^{-2\tau_0(\xi)} = 0 \qquad (\eta=0, |\xi|>1)$$

$$\Theta_0(\xi) = \begin{cases} -\alpha & (\eta=0, c_0<\xi<1) \\ \pi - \alpha & (\eta=0, -1<\xi<c_0) \end{cases} \qquad (3.14)$$

It is noted that the kinematic free-surface condition is automatically satisfied where the free surface on the physical plane is mapped onto the real axis in the segments $|\xi| > 1$ of the ζ-plane. Equation (3.14) gives:

$$\tau_0(\xi) = 0 \qquad (\eta=0, |\xi|>1)$$

So

$$|W_0| = 1 \qquad (\eta=0, |\xi|>1) \qquad (3.15a)$$

And
$$\Theta_0 = \begin{cases} -\alpha & (\eta=0, c_0<\xi<1) \\ \pi - \alpha & (\eta=0, -1<\xi<c_0) \end{cases} \tag{3.15b}$$

At the stagnation point

$$W_0(\, c_0 \,) = 0 \tag{3.15c}$$

The W_0-plane is shown in Figure 3.2. The entire fluid region is mapped by $W_0(Z(\zeta))$ into the region bounded by a unit half-circle and its diameter. The diameter is the image of the planing surface, on which the direction of the velocity vector is known, and the semi-circle is the image of the entire free surface, on which the magnitude of the velocity vector is known. Now, map the W_0-plane onto the lower half of the ζ-plane by either of the following equivalent forms:

$$\begin{aligned} W_0(Z(\zeta)) &= e^{i\alpha} \frac{\zeta-c_0}{1-c_0\zeta+i\sqrt{(1-c_0{}^2)(\zeta^2-1)}} \\ &= e^{i\alpha} \frac{1-c_0\zeta-i\sqrt{(1-c_0{}^2)(\zeta^2-1)}}{\zeta-c_0} \end{aligned} \tag{3.16}$$

$\sqrt{1-\zeta^2}$ is defined as the branch of $(1-\zeta^2)^{1/2}$ with its branch cut on the real ζ-axis between $\zeta=+1,-1$.

Equations (3.9), (3.11) and (3.16) give:

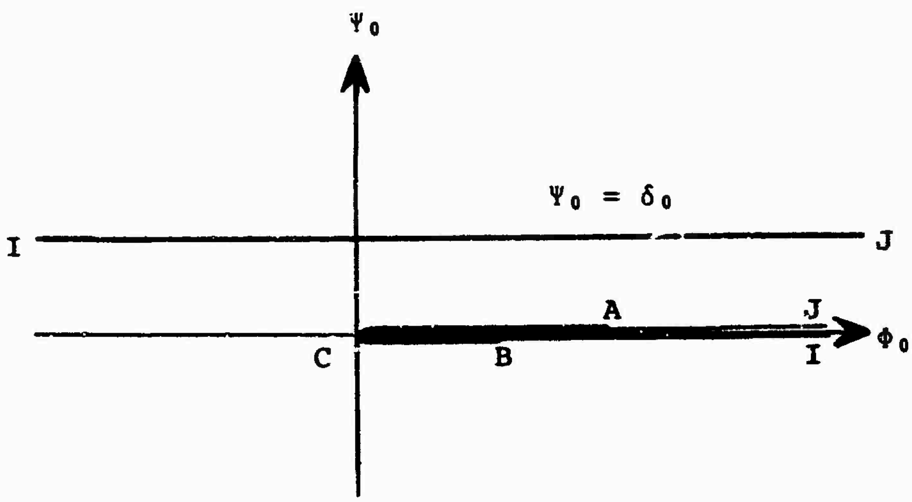

Figure 3.1 Complex Potential F_0-Plane

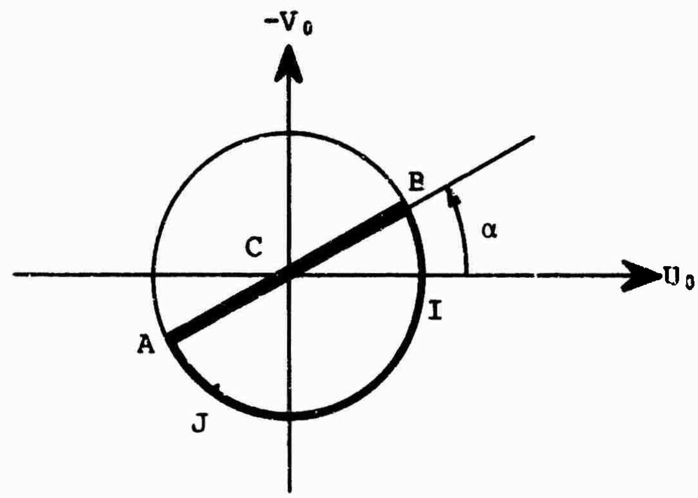

Figure 3.2 Complex Velocity W_0-Plane

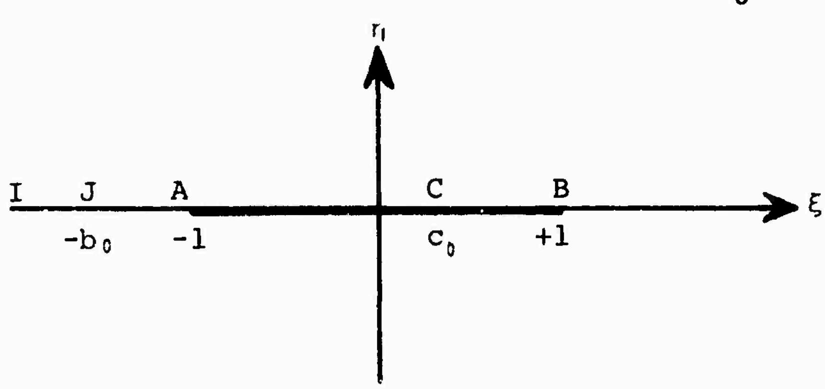

Figure 3.3 ζ-Plane

$$Z(\zeta) = \frac{\delta_0 e^{-i\alpha}}{\pi(b_0+c_0)} \{-c_0(\zeta-1)+(1+c_0b_0)\log\frac{\zeta+b_0}{b_0+1}$$
$$+ i\sqrt{1-c_0^2}\sqrt{\zeta^2-1} - ib_0\sqrt{1-c_0^2}\log(\zeta+\sqrt{\zeta^2-1})$$
$$- i\sqrt{b_0^2-1}\sqrt{1-c_0^2}\log\frac{1+b_0\zeta-\sqrt{(b_0^2-1)(\zeta^2-1)}}{\zeta+b_0}\} \qquad (3.17)$$

This is the inversion formula which gives Z as a function of ζ.

There are three parameters a_0 , b_0 , and c_0 at each section, none of which has been determined yet aside from the facts that

$$|c_0| < 1 \ ; \quad b_0 > 1$$

Equations (3.8) and (3.17) provide a complete first-order inner solution. The three unknown parameters will be determined from matching to the far field problem.

III. 2. First-Order Outer Solution

Consider the flow field at a great distance from the planing surface, mathematically, a distance of order unity with respect to ε. The boundary-value problem has been formulated in section II. 3. With the same kind of argument as given in section II.4, the first approximation for the outer problem is

$$\text{(L)} \qquad \phi_{0xx} + \phi_{0yy} + \phi_{0zz} = 0 \qquad \text{in fluid} \qquad (3.18)$$

$$\text{(F1)} \qquad \phi_{0x}^{\;2} + \phi_{0y}^{\;2} + \phi_{0z}^{\;2} = 1 \qquad \text{on } S_f \qquad (3.19)$$

$$\text{(F2)} \qquad \phi_{0x}\eta_{0x} - \phi_{0y} + \phi_{0z}\eta_{0z} = 0 \qquad \text{on } S_f \qquad (3.20)$$

$$\text{(R)} \qquad \phi_{0x} = 1 \; ; \quad \phi_{0y} = \phi_{0z} = 0 \qquad \begin{matrix} x \to -\infty \\ y \to -\infty \end{matrix} \qquad (3.21)$$

The free-surface boundary conditions (F1) and (F2) are applied on S_f which is unknown and is a part of the problem to be solved. To avoid this difficulty, it is assumed that Taylor series expansions exist for ϕ_0 at the neighborhood of η_0. From (2.12), S_f is

$$y = \eta(x,z) = \eta_0 + \eta_1 + \eta_2 + \; ....$$

It is then assumed, for example, that:

$$\phi_{0x}(x,\eta(x,z),z) = \phi_{0x}(x,\eta_0,z) + (\eta_1+\eta_2+ \;...)\phi_{0xy}(x,\eta_0,z)$$
$$+ \frac{(\eta_1+\eta_2+...)^2}{2} \phi_{0xyy}(x,\eta_0,z) + \; ....$$

However, it will be shown in the later part of this

section that the first term of the outer solution represents a uniform stream. It gives the value of η_0 to be zero. Therefore, it is assumed here that $\eta_0=0$. To the first-order the free-surface conditions can be applied on the undisturbed free surface y=0 to give:

$$\text{(F1)} \qquad \phi_{0x}^{\ 2} + \phi_{0y}^{\ 2} + \phi_{0z}^{\ 2} = 1 \qquad \text{on } y=0 \tag{3.22}$$

$$\text{(F2)} \qquad \phi_{0y} = 0 \qquad \text{on } y=0 \tag{3.23}$$

For a high-aspect ratio planing surface, the outer problem corresponds to a planing surface of fixed span and vanishingly small chord length. As AR $\to \infty$, the planing surface shrinks to a line. There is no disturbance far upstream. Therefore, it is plausible to assume that the outer problem can be linearized. This is confirmed in experiments[30] which show that at a small distance ahead of the planing body the water surface is practically undisturbed.

This physical argument shows that the first order solution is a uniform stream.

$$\phi_0(x,\ y,\ z) = x \tag{3.24}$$

Mathematically, it can be shown that the equation (3.24)

satisfies all the first-order equations, (3.18) - (3.23) trivially.

III. 3. First-Order Matching

In the inner problem, the radiation condition is not considered. On the other hand, the body condition is not considered for the far field solution. These missing conditions can be recovered through the matching process, leading to a unified solution valid in the whole fluid region.

Let m=1 and n=1 in the equation (2.6);

1-term inner expansion of (1-term outer expansion)
= 1-term outer expansion of (1-term inner expansion) (3.25)

Each of these will now be obtained.

Consider the inversion formula. As $\zeta \to \infty$, equation (3.17) gives:

$$Z \sim \frac{\delta_0}{\pi(b_0+c_0)}\,\zeta \qquad \text{as } |Z| \text{ and } |\zeta| \to \infty \qquad (3.26)$$

This equation states that Z and ζ correspond to within a real scale factor at large distances from the planing surface. This has a significant meaning. In order to obtain the outer limit of the inner solution, one has to

work with a large value of Z which corresponds to the large value of ζ as expressed in (3.26).

Let

$$A_0 = \frac{\delta_0}{\pi(b_0+c_0)} \tag{3.27}$$

Far away from the planing surface, which corresponds to the outer limit of the inner problem, equation (3.17) gives: (see Appendix A)

$$\begin{aligned} Z \sim A_0 e^{-i\alpha}\{ & (-c_0+i\sqrt{1-c_0^2})\zeta + \log\zeta(1+c_0b_0-ib_0\sqrt{1-c_0^2}) \\ & - i\sqrt{1-c_0^2}(b_0\log 2 + \sqrt{b_0^2-1}\log(b_0-\sqrt{b_0^2-1})) - c_0 \\ & - (1+c_0b_0)\log(1+b_0)\} + O(\tfrac{1}{\zeta}) \end{aligned} \tag{3.28a}$$

Recall that log ζ is treated as O(1) as ζ approaches a large value. To the first term, this expression can be inverted, as follows:

$$\zeta = \frac{e^{i\alpha}}{A_0(-c_0+i\sqrt{1-c_0^2})} Z + O(\log Z) \tag{3.28b}$$

Now consider the one-term inner expansion. Equation (3.8) gives:

$$\Phi_0 = Re\ \{\frac{\delta_0}{\pi(b_0+c_0)}[\zeta - c_0 - (b_0+c_0)\log\frac{\zeta+b_0}{b_0+c_0}]\}$$

Far away from the planing surface, if (3.28) is used, this becomes:

$$\Phi_0 \sim (-c_0\cos\alpha + \sqrt{1-c_0^{\,2}}\sin\alpha)X + (c_0\sin\alpha + \sqrt{1-c_0^{\,2}}\cos\alpha)Y$$
$$+ O(\log X) \qquad\qquad \text{for large } |Z|$$

In the outer variables, it gives

$$\Phi_0 \sim (-c_0\cos\alpha + \sqrt{1-c_0^{\,2}}\sin\alpha)\frac{x+h\cot\alpha}{\varepsilon}$$
$$+ (c_0\sin\alpha + \sqrt{1-c_0^{\,2}}\cos\alpha)\frac{y-h}{\varepsilon}$$
$$+ O(\log\frac{x+h\cot\alpha}{\varepsilon})$$

where $h = O(\ell) = O(\varepsilon)$

The one-term outer limit of the one-term inner solution is

$$\Phi_0 \sim \frac{1}{\varepsilon}[(-c_0\cos\alpha + \sqrt{1-c_0^{\,2}}\sin\alpha)x + (c_0\sin\alpha + \sqrt{1-c_0^{\,2}}\cos\alpha)y] \qquad (3.29)$$

Next consider the inner limit of the outer solution. The one-term outer solution is

$$\phi_0\ (x, y, z) \sim x$$

In terms of inner variables, this becomes:

$$\phi_0 \sim \varepsilon X - \varepsilon H \cot \alpha$$

This is the one-term inner expansion of the one-term outer solution. To be matched to the expression in (3.29), it must be expressed in the same variables. Thus the one-term inner expansion of the one-term outer expansion is:

$$\phi_0 \, (x, y, z) \sim x \tag{3.30}$$

Equations (3.25), (3.29) and (3.30) give:

$$c_0 = - \cos \alpha \tag{3.31}$$

This was known to Green;[7] however he *assumed* it to be true.

IV. SECOND-ORDER SOLUTION

IV. 1. Second-Order Outer Solution

The boundary-value problem for the second-order outer solution can be obtained from equations (2.13) - (2.16)

(L) $\phi_{1xx} + \phi_{1yy} + \phi_{1zz} = 0$ in fluid (4.1)

(F1) $2(\phi_{0x}\phi_{1x} + \phi_{0y}\phi_{1y} + \phi_{0z}\phi_{1z}) + 2\eta_1(\phi_{0xy}\phi_{0x}$
$+ \phi_{0yy}\phi_{0y} + \phi_{0zy}\phi_{0z}) = 0$ on y=0 (4.2)

(F2) $\phi_{0x}\eta_{1x} + \phi_{1x}\eta_{0x} - \phi_{1y} + \phi_{0z}\eta_{1z} + \phi_{1z}\eta_{0z}$
$+ \eta_1\phi_{0xy}\eta_{0x} - \eta_1\phi_{0yy} + \eta_1\phi_{0zy}\eta_{0z} = 0$
on y=0 (4.3)

(R) $\phi_{1x} = 0$; $\phi_{1y} = \phi_{1z} = 0$ $x \to -\infty$, $y \to -\infty$ (4.4)

If the equation (3.24) is used, the free-surface boundary conditions become

(F1') $\phi_{1x}(x,y,z) = 0$ on y=0 (4.2')

(F2') $\phi_{0x}\eta_{1x} + \phi_{1x}\eta_{0x} - \phi_{1y} + \phi_{1z}\eta_{0z} = 0$
on y=0 (4.3')

The equations (4.1), (4.2'), (4.3') and (4.4) constitute the second order boundary value problem.

The equation (4.2') states that the velocity potential ϕ_1 can be continued analytically into the upper half space as an odd function with respect to y=0. The equation (4.1) states that the potential ϕ_1 is a harmonic function. Upstream of the planing surface, the potential ϕ_1 can be required to satisfy:

$$\phi_1 (x, 0, z) = 0 \qquad \text{on } y=0,\ x<0, \qquad (4.5)$$

In this domain, ϕ_1 must be continuous at y=0. The function ϕ_1 is thus defined in the entire space and has the following property

$$\phi_1 (x, y, z) = - \phi_1 (x, -y, z) \qquad (4.6)$$

It is this result which permits the outer problem of the planing surface to be reduced to an equivalent airfoil problem. Downstream of the planing surface, equation (4.2') then becomes just the usual downstream condition for continuity of pressure across a vortex wake. The body condition is replaced by requiring the outer expansion to match with the inner problem. The line of singularities is now located on the undisturbed free surface, and the

vortex sheet is assumed to lie on the xz-plane which corresponds to the free surface at upstream infinity.

The planing problem will thus be solved as if there were a vortex wake. Actually, there is none; instead there is a free-surface. It is simply that the outer problem of the planing surface is mathematically equivalent to a well-studied airfoil problem. Concepts such as circulation have no place in the planing problem.[25] The fluid region is simply connected and the potential function is single valued. There is no circulation. Nevertheless, some of the terms and symbols of aerodynamics will be used in solving the problem.

The planing surface can be represented by a line of singularities at x=0 along the z-axis in the unbounded fluid as ε approaches zero. The strength of these singularities is still unknown. Since there exists a lift on the body, one must provide for the existence of singularities in the flow behind the body. These are the usual wake vortices, which can be represented in terms of a dipole distribution of density $\gamma(x,z)$ in the xz-plane, with doublet axes normal to y=0 plane.

The wake now is a surface of discontinuity; therefore the distribution of doublets must be extended to infinity over the wake surface W as well as over the planing surface S. The span of the planing surface is located in the segment between z=-1,+1. The velocity potential ϕ_1 has the following form[9]

$$\phi_1(x,y,z) = \frac{1}{4\pi}\int_{-1}^{1} d\zeta \int_{0}^{\infty} d\xi \; \frac{y\gamma(\zeta)}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}}$$

$$+ \frac{1}{4\pi}\int_{-1}^{1} \frac{y\mu(\zeta)d\zeta}{[x^2+y^2+(z-\zeta)^2]^{3/2}} + \frac{1}{4\pi}\int_{-1}^{1} \frac{x\lambda(\zeta)d\zeta}{[x^2+y^2+(z-\zeta)^2]^{3/2}} \tag{4.7}$$

It can be shown that the equation (4.7) satisfies the Laplace equation and all the boundary conditions (4.1) - (4.4). The first integral represents a plane sheet of dipoles extending to infinity downstream. As shown in Appendix B, the dipole density is a function of the z variable only, $\gamma=\gamma(z)$. The second and third integrals represent discrete lines of vertical and horizontal dipoles, respectively. Higher order singularities, such as quadripoles, etc., need not be considered here for the second-order approximation. Sources have been omitted, on the assumption that there is no generation of fluid in the body.

The first integral can be integrated with respect to ξ to give

$$\phi_1(x,y,z) = \frac{y}{4\pi}\int_{-1}^{1} \frac{\gamma(\zeta)}{y^2+(z-\zeta)^2}\left[1+ \frac{x}{[x^2+y^2+(z-\zeta)^2]^{1/2}}\right]d\zeta$$

$$+ \frac{y}{4\pi}\int_{-1}^{1} \frac{\mu(\zeta)d\zeta}{[x^2+y^2+(z-\zeta)^2]^{3/2}}$$

$$+ \frac{x}{4\pi}\int_{-1}^{1} \frac{\lambda(\zeta)d\zeta}{[x^2+y^2+(z-\zeta)^2]^{3/2}} \tag{4.8}$$

The functions $\gamma(z)$, $\lambda(z)$, and $\mu(z)$ can be expanded into series which are asymptotically valid as $\varepsilon \to 0$

$$\gamma(z) \sim \sum_{n=1}^{N} \gamma_n(z;\varepsilon) \qquad \gamma_{n+1} = o(\gamma_n) \text{ as } \varepsilon \to 0$$

$$\mu(z) \sim \sum_{n=1}^{N} \mu_n(z;\varepsilon) \qquad \mu_{n+1} = o(\mu_n) \text{ as } \varepsilon \to 0$$

$$\lambda(z) \sim \sum_{n=1}^{N} \lambda_n(z;\varepsilon) \qquad \lambda_{n+1} = o(\lambda_n) \text{ as } \varepsilon \to 0 \tag{4.9}$$

In the outer problem, as aspect-ratio AR approaches infinity, a uniform stream is recovered, and the dipole density eventually vanishes. Therefore, one has

$$\begin{aligned} \gamma_1(z) &= o\ (1) \\ \mu_1(z) &= o\ (1) \\ \lambda_1(z) &= o\ (1) \end{aligned} \tag{4.10}$$

Therefore, the two-term outer solution is

$$\phi_0 + \phi_0 \sim x + \frac{y}{4\pi}\int_{-1}^{1} \frac{\gamma_1(\zeta)}{y^2+(z-\zeta)^2}\left[1 + \frac{x}{[x^2+y^2+(z-\zeta)^2]^{1/2}}\right] d\zeta$$

$$+ \frac{y}{4\pi}\int_{-1}^{1} \frac{\mu_1(\zeta)}{[x^2+y^2+(z-\zeta)^2]^{3/2}}\, d\zeta$$

$$+ \frac{x}{4\pi}\int_{-1}^{1} \frac{\lambda_1(\zeta)}{[x^2+y^2+(z-\zeta)^2]^{3/2}}\, d\zeta \qquad (4.11)$$

IV. 2. Intermediate Step Matching

Let m= 1 and n=2 in the equation (2.6) to give:

2-t outer expansion of (1-t inner expansion)

= 1-t inner expansion of (2-t outer expansion)

(4.12)

In order to find the former, procede as follows:
For large ζ, the inversion formula (3.17) gives a two-term expansion:

$$Z \sim A_0\zeta + A_0(e^{-i\alpha} - b_0)\log\zeta$$
$$+ A_0e^{-i\alpha}(M_0 - iR_0) + O(\tfrac{1}{\zeta}) \qquad \text{for large } \zeta \qquad (4.13a)$$

where

$$M_0 = c_0 - (1+b_0c_0)\log(1+b_0)$$
$$R_0 = b_0\sqrt{1-c_0^2}\log 2 + \sqrt{b_0^2-1}\sqrt{1-c_0^2}\log(b_0-\sqrt{b_0^2-1})$$

(4.13b)

Next, express ζ in terms of Z to give

$$\zeta = \frac{Z}{A_0} - (e^{-i\alpha} - b_0) \log \frac{Z}{A_0} - (M_0 - iR_0)e^{-i\alpha} + O(\frac{1}{Z})$$

(4.13c)

one-term inner solution is given by equation (3.8). Far away from the planing surface, equation (3.8) gives:

$$\begin{aligned}\Phi_0 \sim Re \; \{A_0[\frac{Z}{A_0} - (e^{-i\alpha} - b_0) \log\frac{Z}{A_0} - e^{-i\alpha}(M_0 - iR_0) \\ - c_0 - (b_0+c_0) \log\frac{Z}{A_0} + (b_0+c_0) \log(b_0+c_0) \\ + O(\frac{1}{Z})]\}\end{aligned}$$

where A_0, M_0, and R_0 are all real constants. Next take the real part for large value of X to give

$$\Phi_0 \sim X - A_0 \sin\alpha \cdot \tan^{-1} \frac{Y}{X} - G_0 + O(\frac{1}{X})$$

The two-term outer limit of the one-term inner expansion is thus

$$\Phi_0 \sim X - A_0 \sin\alpha \cdot \tan^{-1} \frac{Y}{X} - G_0 \tag{4.14a}$$

where

$$\begin{aligned}G_0 = A_0[M_0 \cos\alpha - R_0 \sin\alpha - \cos\alpha \\ -(b_0 - \cos\alpha) \log(b_0 - \cos\alpha)]\end{aligned} \tag{4.14b}$$

Take outer variables

$$\Phi_0 \sim \frac{x+h\cot\alpha}{\varepsilon} - A_0 \sin\alpha\tan^{-1}\frac{y-h}{x+h\cot\alpha} - G_0$$

$$\sim \frac{1}{\varepsilon}(x - \varepsilon A_0\sin\alpha \tan^{-1} \frac{y}{x} + h \cot\alpha - \varepsilon G_0)$$

A constant K_0, which is independent of space coordinates, added or subtracted from a velocity potential would not violate the conditions prescirbed for Φ_0 (Laplace equation and boundary conditions). So the two-term outer expansion of the one-term inner expansion is:

$$\Phi_0 \sim \frac{1}{\varepsilon} (x - \varepsilon A_0\sin\alpha \cdot \tan^{-1} \frac{y}{x} + h \cot\alpha - \varepsilon G_0 - \varepsilon K_0) \tag{4.15}$$

The two-term outer expansion is given in (4.11). To take the inner limit, this solution must be evaluated near the line of singularities. Then the integrals in (4.11) become divergent integrals. This difficulty is resolved by integration by part to give (see Appendix C)

$$\phi_0 + \phi_1 \sim x - \frac{1}{4\pi}\int_{-1}^{1} (\tan^{-1}\frac{y}{z-\zeta} + \tan^{-1}\frac{y}{x}\cdot\frac{\sqrt{x^2+y^2+(z-\zeta)^2}}{z-\zeta})\frac{d\gamma_1}{d\zeta}d\zeta$$

$$+ \frac{1}{4\pi}\,\frac{y}{x^2+y^2}\int_{-1}^{1} \frac{(z-\zeta)}{[x^2+y^2+(z-\zeta)^2]^{1/2}}\,\frac{d\mu_1}{d\zeta}\, d\zeta$$

$$+ \frac{1}{4\pi} \cdot \frac{x}{x^2+y^2} \int_{-1}^{1} \frac{(z-\zeta)}{[x^2+y^2+(z-\zeta)^2]^{1/2}} \cdot \frac{d\lambda_1}{d\zeta} \, d\zeta \qquad (4.16)$$

The strength of singularities is zero outside the planing body, which gives:

$$\gamma_1(\pm 1) = 0 \quad ; \quad \mu_1(\pm 1) = 0 \quad ; \quad \lambda_1(\pm 1) = 0$$

Take inner variables

$$\begin{aligned} \phi_0 + \phi_1 \sim \; & (\varepsilon X - \varepsilon H \cot \alpha) - \frac{1}{4\pi}(\varepsilon Y + \varepsilon H) \int_{-1}^{1} \frac{d\gamma_1}{d\zeta} \frac{d\zeta}{z-\zeta} \\ & - \frac{\gamma_1(z)}{2\pi} \tan^{-1}\left(\frac{\varepsilon Y + \varepsilon H}{\varepsilon X - \varepsilon H \cot \alpha}\right) \\ & + \frac{1}{2\pi} \frac{(\varepsilon Y + \varepsilon H)\mu_1(z) + (\varepsilon X - \varepsilon H \cot \alpha)\lambda_1(Z)}{(\varepsilon X - \varepsilon H \cot \alpha)^2 + (\varepsilon Y + \varepsilon H)^2} + o(\varepsilon \gamma_1) \end{aligned} \qquad (4.17)$$

The Cauchy Principal value is to be taken for the integral.

The order of magnitude of $\gamma_1(z)$ with respect to ε is not known yet. However, one does know that $\gamma_1(z) = o(1)$. Therefore, the one-term inner expansion of the two-term outer expansion is:

$$\begin{aligned} \phi_0 + \phi_1 \sim (\varepsilon X - \varepsilon H \cot \alpha) & - \frac{\gamma_1(Z)}{2\pi} \tan^{-1}\left(\frac{\varepsilon Y + \varepsilon H}{\varepsilon X - \varepsilon H \cot \alpha}\right) \\ & + \frac{1}{2\pi} \frac{(\varepsilon Y + \varepsilon H)\mu_1(Z) + (\varepsilon Y - \varepsilon H \cot \alpha)\lambda_1(Z)}{(\varepsilon X - \varepsilon H \cot \alpha)^2 + (\varepsilon Y + \varepsilon H)^2} \end{aligned}$$

In the outer variables, this is :

$$\phi_0 + \phi_1 \sim x - \frac{\gamma_1(z)}{2\pi} \tan^{-1}\left(\frac{y}{x}\right) + \frac{y}{2\pi}\mu_1(z) + \frac{x}{2\pi}\lambda_1(z) \qquad (4.18)$$

From the equations (4.12), (4.15) and (4.18), matching gives

$$\gamma_1(z) = 2\pi\varepsilon A_0 \sin\alpha$$

$$\mu_1(z) = 0$$

$$\lambda_1(z) = 0$$

$$G_0 + K_0 - h\cot\alpha = 0 \qquad (4.19)$$

The discrete lines of vertical and horizontal dipoles $\mu_1(z)$ and $\lambda_1(z)$ are of higher-order than that of $\gamma_1(z)$. So

$$\gamma_1(z) = O(\varepsilon)$$

$$\mu_1(z) = \lambda_1(z) = o(\varepsilon)$$

$$\phi_1(x,y,z) = O(\varepsilon) \qquad (4.20)$$

IV. 3. Second-Order Inner Solution

From the first-order outer solution, $\Phi_0 = O(\frac{1}{\varepsilon})$ as expected. However, the definition of the asymptotic expansion only implies that $\Phi_1 = o(\Phi_0)$. The exact order of magnitude of Φ_1 is still unknown. Therefore, $\varepsilon^2\Phi_{0ZZ}$ term must be included in the Laplace equation.

Next, use the same argument given in section II. 4. The possible second-order problems obtained from the equations (2.23) - (2.26) are included in the following:

$$\text{(L)} \qquad \Phi_{1XX} + \Phi_{1YY} + \varepsilon^2\Phi_{0ZZ} \sim 0 \qquad (4.21a)$$

$$\text{(F1)} \qquad 2\Phi_{0X}\Phi_{1X} + 2\Phi_{0Y}\Phi_{1Y} + \varepsilon^2\Phi_{0Z}{}^2 \sim 0 \qquad (4.21b)$$

$$\text{(F2)} \qquad \Phi_{0X}N_{1X} + \Phi_{1X}N_{0X} - \varepsilon\Phi_{1Y} + \varepsilon^2\Phi_{0Z}N_{0Z} \sim 0 \qquad (4.21c)$$

$$\text{(B)} \qquad \Phi_{1X}\sin\alpha + \Phi_{1Y}\cos\alpha = 0 \qquad (4.21d)$$

Now, the order of Φ_1 will be estimated. There are two possible ways to do this. One is to assign every possible order of magnitude to Φ_1, then try to show that each assigned order of magnitude, except one, gives rise to a contradition among the Laplace equation and all the available boundary conditions. The exceptional case which is acceptable for each equation is thus the possible order of magnitude for Φ_1. The second approach is to consider all the non-homogeneous equations. Determine the order of magnitude of non-homogeneous parts. The lowest one will be the possible order of Φ_1. The nice thing about the method of matched expansions is that, if the order of magnitude is not correctly estimated, it

will show up automatically in the later part during the matching performance. The second method will be used here to estimate Φ_1.

The available non-homogeneous equations are

$$\underset{O(\Phi_1)}{\Phi_{1XX}} + \underset{O(\Phi_1)}{\Phi_{1YY}} = \underset{O(\varepsilon)}{-\varepsilon^2\Phi_{0ZZ}} \tag{4.22a}$$

$$\underset{O(\frac{\Phi_1}{\varepsilon})}{2\Phi_{0X}\Phi_{1X}} + \underset{O(\frac{\Phi_1}{\varepsilon})}{2\Phi_{0Y}\Phi_{1Y}} = \underset{O(1)}{-\varepsilon^2\Phi_{0Z}^{\ 2}} \tag{4.22b}$$

$$\Phi_{0X}N_{1X} + \Phi_{1X}N_{0X} - \varepsilon\Phi_{1Y} = -\varepsilon^2\Phi_{0Z}N_{0Z} \tag{4.22c}$$

Equation (4.17) gives:

$$\text{1-T. I. E. of 2-T. O. E.} \sim O(\varepsilon X) \tag{4.22d}$$

and

$$\text{2-T. I. E. of 2-T. O. E.} \sim O(\varepsilon X) + O(\varepsilon^2 Y) \tag{4.22e}$$

The first term in the right hand side of (4.22e) is matched to inner solution Φ_0 and the second term is matched to Φ_1. Therefore, the equation (4.22e) gives

$$\Phi_1 \sim O(\varepsilon\Phi_0) \tag{4.22f}$$

The kinematic free-surface condition (4.22c) will be used to determine the free-surface deflection, therefore, it can only be used to determine the order of N_1. From (4.22a), (4.22b), and (4.22f), the lowest order for Φ_1 is

$$\Phi_1 = O(1) \tag{4.23}$$

If equation (4.23) is used, the second-order inner problem becomes

$$\text{(L)} \qquad \Phi_{1XX} + \Phi_{1YY} = 0 \qquad \text{in fluid} \tag{4.24a}$$

$$\text{(F1)} \qquad 2(\Phi_{0X}\Phi_{1X} + \Phi_{0Y}\Phi_{1Y}) = 0 \qquad \text{on } S_f \tag{4.24b}$$

$$\text{(F2)} \qquad \Phi_{0X}N_{1X} + \Phi_{1X}N_{0X} - \varepsilon\Phi_{1Y} = 0 \qquad \text{on } S_f \tag{4.24c}$$

$$\text{(B)} \qquad \Phi_{1X}\sin\alpha + \Phi_{1Y}\cos\alpha = 0 \qquad \text{on body} \tag{4.24d}$$

Instead of this boundary-value problem being solved directly, it will be combined with the first-order inner problem. The new boundary-value problem will be shown to have exactly the same forms as the first-order inner problem, and the solution obtained in the previous

chapter can thus be carried over directly to give the second-order inner solution.

Consider the new boundary-value problem in which first-order and second-order conditions are added:

$$\Phi_{0XX} + \Phi_{0YY} + \Phi_{1XX} + \Phi_{1YY} = 0 \quad \text{in fluid} \tag{4.25a}$$

$$\Phi_{0X}^2 + \Phi_{0Y}^2 + 2(\Phi_{0X}\Phi_{1X} + \Phi_{0Y}\Phi_{1Y}) = 1 \quad \text{on } S_f \tag{4.25b}$$

$$\Phi_{0X}N_{0X} - \varepsilon\Phi_{0Y} + \Phi_{0X}N_{1X} + \Phi_{1X}N_{0X} - \varepsilon\Phi_{1Y} = 0 \quad \text{on } S_f \tag{4.25c}$$

$$\Phi_{0X}\sin\alpha + \Phi_{0Y}\cos\alpha + \Phi_{1X}\sin\alpha + \Phi_{1Y}\cos\alpha = 0 \quad \text{on body} \tag{4.25d}$$

Now, consider the equation (4.25b). If Φ_{1X}^2 and Φ_{1Y}^2 are added to the left side of equation (4.25b), the effect will not be felt until the third approximation or higher. Therefore, without loss of accuracy to the second-order approximation, Φ_{1X}^2 and Φ_{1Y}^2 can be added onto (4.25b); the same argument is applied on (4.25b) and (4.25c). One obtains:

$$\text{(L)} \qquad \Phi^*_{XX} + \Phi^*_{YY} = 0 \quad \text{in fluid} \tag{4.26a}$$

$$\text{(F1)} \qquad \Phi^{*2}_{X} + \Phi^{*2}_{Y} = 1 \quad \text{on } S_f \tag{4.26b}$$

$$\text{(F2)} \qquad \Phi^*_X N^*_X - \epsilon\Phi^*_Y = 0 \qquad \text{on } S_\epsilon \qquad (4.26c)$$

$$\text{(B)} \qquad \Phi^*_X \sin\alpha + \Phi^*_Y \cos\alpha = 0 \qquad \text{on body} \qquad (4.26d)$$

where

$$\Phi^* = \Phi_0 + \Phi_1$$

$$N^* = N_0 + N_1 \qquad (4.27)$$

This is the new boundary-value problem. The governing differential equation is the two dimensional Laplace equation. All the boundary conditions are two dimensional. Therefore, the solution represents a plane irrotational flow. The method developed in the first-order inner solution can be applied directly to this new problem. Once Φ^* is determined, the second-order potential Φ_1 can be obtained easily.

In the first-order problem, a_0, b_0 and c_0 represent the jet thickness, jet direction and the location of stagnation point on the ζ-plane respectively. Now due to the three dimensional effect, these values will be changed by a small amount.

Let a, b, and c represent true values, assume the existence of asymptotic expansions for a, b, and c

$$a \sim \sum_{n=0}^{N} a_n \qquad a_{n+1} = o(a_n) \text{ as } \epsilon \to 0 \qquad (4.28a)$$

$$b \sim \sum_{n=0}^{N} b_n \qquad b_{n+1}=o(b_n) \text{ as } \varepsilon \to 0 \tag{4.28b}$$

$$c \sim \sum_{n=0}^{N} c_n \qquad c_{n+1}=o(c_n) \text{ as } \varepsilon \to 0 \tag{4.28c}$$

For the new boundary-value problem Φ^*, let

$$\begin{aligned} a^* &= a_0 + a_1 \\ b^* &= b_0 + b_1 \\ c^* &= c_0 + c_1 \end{aligned} \tag{4.29}$$

Since Φ^* is a harmonic function, its conjugate, Ψ^*, exists. Let the analytic function F^* be the complex potential

$$F^*(z) = F_0(Z) + F_1(Z) = \Phi^*(X,Y) + i\Psi^*(X,Y) \tag{4.30}$$

Use the same techniques as given in section III. 1. Map the F^*-plane onto the lower half of ζ-plane as shown in figures 3.1 and 3.2 to give:

$$F^*(Z) = \frac{\delta^*}{\pi(b^*+c^*)}\left[\zeta - c^* - (b^*+c^*)\log\frac{\zeta+b^*}{b^*+c^*}\right] \tag{4.31}$$

Next, define the new mapping function to be $H^*(\zeta)$. Then the

inversion formula is:

$$Z = \int_1^{\zeta} \frac{H^*(\zeta')}{W^*(Z(\zeta'))}\, d\zeta' \qquad (4.32)$$

where $W^*(Z)$ is the complex velocity.

Next, define the Kirchhoff function Q^*:

$$Q^* = Q_0 + Q_1 = (\tau_0+\tau_1) + i(\theta_0+\theta_1) = -\log W^* \qquad (4.33)$$

Then on the free surface it gives:

$$1 - e^{-2(\tau_0+\tau_1)} = 0 \qquad \text{on } S_f \qquad (4.34a)$$

On the body:

$$\theta_0 + \theta_1 = -\alpha \qquad \text{on } \overline{CB}$$

$$\theta_0 + \theta_1 = \pi - \alpha \qquad \text{on } \overline{AC} \qquad (4.34b)$$

Now, map these boundaries onto the real axis, $\eta=0$, of the ζ-plane. The conditions become:

$$1 - e^{-2(\tau_0(\xi) + \tau_1(\xi))} = 0 \qquad (\eta=0, |\xi|>1) \qquad (4.35a)$$

$$\theta_0 + \theta_1 = \begin{cases} -\alpha & (\eta=0, c_0+c_1<\xi<1) \\ \pi-\alpha & (\eta=0, -1<\xi<c_0+c_1) \end{cases} \qquad (4.35b)$$

Equation (4.35) gives:

$$W^*(Z(\zeta)) = e^{i\alpha} \frac{\zeta - (c_0+c_1)}{1 - (c_0+c_1)\zeta + i\sqrt{[1-(c_0+c_1)^2](\zeta^2-1)}} \tag{4.36}$$

Then the inversion formula gives:

$$\begin{aligned} Z = \frac{\delta^* e^{-i\alpha}}{\pi(b^*+c^*)} \{ & -c^*(\zeta-1) + (1+c^*b^*) \log \frac{\zeta+b^*}{b^*+1} \\ & + i\sqrt{1-c^{*2}}\sqrt{\zeta^2-1} - ib^*\sqrt{1-c^{*2}} \log (\zeta+\sqrt{\zeta^2-1}) \\ & - i\sqrt{b^{*2}-1}\sqrt{1-c^{*2}} \log \frac{1+b^*\zeta - \sqrt{(b^{*2}-1)(\zeta^2-1)}}{\zeta+b^*} \} \end{aligned} \tag{4.37}$$

Equations (4.31) and (4.37) provide a complete second-order inner solution.

IV. 4. Second Order Matching

Let m=2 and n=2 in (2.6), giving:

$$\begin{aligned} & \text{2-T. O. E. of (2-T. I. E.)} \\ & = \text{2-T. I. E. of (2-T. O. E.)} \end{aligned} \tag{4.38}$$

For large ζ, (4.37) gives two-term expression.

$$\begin{aligned} Z = & \left(A_0+A_1+iA_0\frac{c_1}{\sin\alpha}\right)\zeta + A_0(e^{-i\alpha} - b_0) \log \zeta \\ & + A_0 e^{-i\alpha}(M_0 - iR_0) \end{aligned} \tag{4.39a}$$

where $$A_1 = \frac{\delta_1}{\pi(b_0+c_0)} - \frac{\delta_0(b_1+c_1)}{\pi(b_0+c_0)^2} \tag{4.39b}$$

and A_0, M_0 , and R_0 are defined in (41.3b).

Next, express ζ in terms of Z to two terms:

$$\zeta = \frac{Z}{A_0} - \frac{Z}{A_0}\left(\frac{A_1}{A_0} + i\frac{c_1}{\sin\alpha}\right) - (e^{-i\alpha} - b_0)\log\frac{Z}{A_0}$$
$$+ A_0 e^{-i\alpha}(iR_0 - M_0) \tag{4.39c}$$

Substitute (4.13b), (4.29), and (4.39b) into (4.31) to give:

$$\begin{aligned}\Phi_0 + \Phi_1 \sim Re\ \{ & (A_0+A_1)\left[\frac{Z}{A_0} - \frac{Z}{A_0}\left(\frac{A_1}{A_0} + i\frac{c_1}{\sin\alpha}\right)\right.\\ & \left. - (e^{-i\alpha}-b_0)\log\frac{Z}{A_0} - iR_0e^{-i\alpha} - M_0e^{-i\alpha}\right]\\ & - (c_0+c_1) - (b_0+b_1+c_0+c_1)\log\frac{Z}{A_0}\\ & + (b_0+c_0+b_1+c_1)\log(b_0+b_1+c_0+c_1)\} + O(\tfrac{1}{Z})\end{aligned} \tag{4.40}$$

Expressed in terms of outer variables, then the two-term outer expansion of the two-term inner expansion is

$$\Phi_0 + \Phi_1 \sim \frac{1}{\varepsilon}\left[x + \frac{c_1}{\sin\alpha}y - \varepsilon A_0 \sin\alpha \tan^{-1}\frac{y}{x} + h\cot\alpha - \varepsilon G_0 - \varepsilon K_0\right] \tag{4.41}$$

Now consider the inner limit of the outer solution. The two-term outer solution is given in equation (4.16). It is assumed that the derivative of $\gamma_1(z)$ with respect to the space coordinate z is the same order of magnitude as $\gamma_1(z)$. Then the two-term inner expansion of the two-term outer expansion is:

$$\phi_0 + \phi_1 \sim x - \frac{\gamma_1(z)}{2\pi} \tan^{-1}\frac{y}{x} - \frac{y}{4\pi}\int_{-1}^{1} \frac{d\gamma_1}{d\zeta}\frac{d\zeta}{z-\zeta}$$

$$+ \frac{1}{2\pi}\frac{y\mu_1(z) + x\lambda_1(z)}{x^2+y^2} \tag{4.42}$$

From (4.38), (4.41) and (4.42), matching gives

$$c_1 = -\frac{\sin\alpha}{4\pi}\int_{-1}^{1} \frac{d\gamma_1}{d\zeta}\frac{d\zeta}{z-\zeta} \tag{4.43}$$

$$\mu_1(z) = \lambda_1(z) = 0 \tag{4.44}$$

The equation (4.44) states that $\mu_1(z)$ and $\lambda_1(z)$ are of higher order. Therefore:

$$\mu_1(z) = \lambda_1(z) = o(\varepsilon) \tag{4.45}$$

From (4.43), c_1 depends on the rate of change of $\gamma_1(z)$ with respect to the spanwise variable. To the first order, equation (3.31) gives

$$\alpha = \cos^{-1}(-c_0)$$

Now the hydrodynamic angle of attack α_H will be

$$\alpha_H = \cos^{-1}(-c_0 - c_1)$$

Therefore, c_1 represents the deviation of α_H from α. So c_1 represents a "downwash", as it is called in aerodynamics.

V. FREE-SURFACE MATCHING

V. 1. Introduction

The planing-surface problem is distinguished from the airfoil problem by the presence of a free-surface. In Green's classic two dimensional problem of a plate gliding on the surface of a gravity-free, infinitely deep fluid, the non-linear free-surface problem can be solved exactly. However, Green's solution exhibits an anomaly. The free-surface elevation is given by

$$y \sim -\log |x| \qquad \text{as } |x| \to \infty \qquad (5.1)$$

Thus far away from the gliding plate, the free-surface drops off logarithmically to $-\infty$. The location of the gliding plate can not be prescribed and his solution is not unique.

The present chapter will show that the anomaly exhibited in this classic solution is just part of a proper near-field expansion of the complete solution which includes three dimensional effects. An inner expansion does not necessarily satisfy the obvious conditions at infinity; it must only match some outer expansion in a proper way. A far-field free-surface expansion will be derived and matched with this inner expansion. Through this free surface matching, the height of the planing surface is properly defined. The depth of submergence can thus be prescribed and the solution becomes unique.

V. 2. First-Order Free-Surface Matching

A. Near-Field Free-Surface Solution

The free surface and planing plate on the physical plane are mapped onto the real axis of the auxiliary complex ζ-plane. The inversion formula (3.17) gives the relationship between the physical Z-plane and the ζ-plane. To describe the free-surface elevation, let $\eta=0$ and $\zeta=\xi$ on the equation (3.17). The image of the free surface $(\eta=0,|\xi|>1)$ in the ζ-plane can now be transformed back to the Z-plane:

$$Z = \int_1^{\xi} \frac{H_0(\xi')}{W_0(Z(\xi'))}\, d\xi' = X(\xi) + iY(\xi) \qquad (5.2)$$

$X(\xi)$ and $Y(\xi)$ are the coordinates of a point in the free surface.

Far away from the planing surface, the equation (5.2) gives:

$$Z = X + iY = A_0\{ \xi + (e^{-i\alpha}-b_0) \log \xi + e^{-i\alpha}(M_0 - iR_0)\} + O(\tfrac{1}{\xi}) \qquad (5.3)$$

A_0, M_0, and R_0 are defined in (4.13b). The leading edge is mapped onto $\zeta=-1$ and the jet onto $\zeta=-b_0$. Therefore, the upstream free surface corresponds to

$$\xi < -b_0 < -1 \qquad (5.4)$$

$$\log \xi = \log |\xi| - \pi i \qquad (5.5)$$

Separate the real and imaginary parts in (5.3) and eliminate

the parameter ξ to give :

$$Y \sim -A_0 \sin\alpha \log|X| + A_0[\sin\alpha \log A_0 - \pi(\cos\alpha - b_0) - M_0 \sin\alpha - R_0 \cos\alpha] + O\left(\log\frac{|X|}{X}\right) \qquad (5.6)$$

This is the free-surface elevation far upstream and downstream from the planing surface. As $X \to \pm\infty$, the free-surface deflection becomes $Y \to -\infty$. This anomaly is also observed in Wagner's and Green's two dimensional solutions. However, it will be shown in the present work that the equation (5.6) is just a proper inner description.

B. Far-Field Free-Surface Solution

Equation (3.20) gives

$$\phi_{0x}\eta_{0x} - \phi_{0y} + \phi_{0z}\eta_{0z} = 0 \qquad \text{on } y=0 \qquad (5.7)$$

If (3.24) is used, equation (5.8) becomes

$$\eta_{0x} = 0$$

Integration gives

$$\eta_0 = D(z) \qquad (5.8)$$

where D is function of z only. However (3.24) represents a uniform stream and, to the same order of magnitude, the free-surface will correspond to the undisturbed free-surface elevation. This gives D=0. So

$$\eta_0 = 0 \qquad (5.9)$$

This will be called the "one-term expansion" of the free-surface elevation.

C. First-Order Free-Surface Matching

Equation (3.29) gives

$$\varepsilon\Phi_0 \sim (-c_0\cos\alpha + \sqrt{1-c_0{}^2}\sin\alpha)x + (c_0\sin\alpha+\sqrt{1-c_0{}^2}\cos\alpha)y$$
$$\sim x \qquad \text{as } |z| \to \infty \qquad (5.10)$$

where $c_0 = -\cos\alpha$ is used. This is the one-term outer expansion of the one-term inner expansion of the velocity potential, which corresponds to a uniform stream at its outer limit. Far away the free-surface can thus be represented by

$$N_0 = 0 \qquad \text{on } S_f \qquad (5.11)$$

This is the one-term outer expansion of the one-term inner expansion of the free-surface elevation.

From (5.9), take inner variables to obtain one-term expression and then converted back to outer variables to give

$$\eta_0 = 0 \qquad (5.12)$$

This is the one-term inner expansion of the one-term outer expansion.

Equations (5.12) and (5.13) agree with the matching principle (2.6). The first-order free-surface matching does not provide additional information; in a sense, it is trivial. Nevertheless this matching does establish the relationship so that the higher-order free-surface expansions will follow

the matching principle.

V. 3. Second-Order Outer Free-Surface Solution

Equation (4.3') gives the kinematic free-surface condition:

$$\phi_{0x}\eta_{1x} + \phi_{1x}\eta_{0x} - \phi_{1y} + \phi_{1z}\eta_{0z} = 0 \quad \text{on } y=0 \qquad (5.13)$$

If (3.24) and (5.9) are used, one obtains:

$$\eta_{1x} - \phi_{1y} = 0 \qquad \text{on } y=0 \qquad (5.14)$$

Integration with respect to x gives

$$\eta_1 = \int_{-\infty}^{x} \phi_{1y}\,(\xi,\ 0,\ z)d\xi \qquad (5.15)$$

The lower limit is taken to be $-\infty$ to coincide with the radiation condition. In appendix D, it is shown that

$$\phi_{1y}(x,0,z) = -\frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1(\zeta)}{d\zeta}\left[\frac{1}{z-\zeta} + \frac{\sqrt{x^2+(z-\zeta)^2}}{x(z-\zeta)}\right] d\zeta \qquad (5.16)$$

where the Cauchy principal value is taken. Substitute (5.16) into (5.15) and change the order of integration to give

$$\eta_1 = -\frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1(\zeta)}{d\zeta}\,\frac{1}{z-\zeta}\Big[x + \sqrt{x^2+(z-\zeta)^2} - |z-\zeta|\log\left|\frac{|z-\zeta| + \sqrt{x^2+(z-\zeta)^2}}{x}\right|\Big]d\zeta \qquad (5.17)$$

The two-term outer solution is thus

$$\eta_0 + \eta_1 = -\frac{1}{4\pi}\int_{-1}^{1} \frac{d\gamma_1(\zeta)}{d\zeta}\frac{1}{z-\zeta}\,[x+\sqrt{x^2+(z-\zeta)^2}$$
$$- |z-\zeta|\,\log\left|\frac{|z-\zeta|+\sqrt{x^2+(z-\zeta)^2}}{x}\right|]d\zeta \qquad (5.18)$$

V. 4. Intermediate Free-Surface Matching

Let m=1 and n=2 in (2.6) to give

1-T. I. E. of (2-T. O. E.)
= 2-T. O. E. of (1-T. I . E.) (5.19)

Substitute inner variables in (5.18) to give

$$\eta_0 + \eta_1 \sim -\frac{1}{2\pi}\gamma_1(Z)\,[1+\log|\varepsilon X-\varepsilon H\cot\,\alpha|]$$
$$+\frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1(\zeta)}{d\zeta}\,\mathrm{sgn}\,(Z-\zeta)\,\log 2|Z-\zeta|d\zeta$$
$$-\frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1(\zeta)}{d\zeta}\,\frac{\varepsilon X-\varepsilon H\cot\,\alpha}{Z-\zeta}d\zeta + o(\varepsilon^2) \qquad (5.20)$$

where $$\mathrm{sgn}\,(Z-\zeta) = \begin{cases} +1 & Z-\zeta>0 \\ -1 & Z-\zeta<0 \end{cases} \qquad (5.21)$$

So the one-term inner expansion of the two-term outer expansion is:

$$\eta_0 + \eta_1 \sim -\frac{1}{2\pi}\gamma_1(z)\,[1+\log|x|]$$
$$+\frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1(\zeta)}{d\zeta}\,\mathrm{sgn}\,(z-\zeta)\,\log 2|z-\zeta|d\zeta \qquad (5.22)$$

Next, substitute outer variables in (5.6) to give the two-term outer expansion of the one-term inner expansion:

$$y \sim h - \varepsilon A_0\sin\,\alpha\log\,|x| + \varepsilon A_0\sin\,\alpha\log\,\varepsilon + \varepsilon E_0 \qquad (5.23)$$

where

$$E_0 = A_0[\sin\alpha \log A_0 - \pi(\cos\alpha - b_0) - M_0 \sin\alpha - R_0 \cos\alpha] \tag{5.24}$$

From (5.19), (5.22) and (5.23), the intermediate matching gives

$$h = -\varepsilon\log \varepsilon A_0 \sin\alpha - \varepsilon E_0 - \frac{1}{2\pi}\gamma_1(z) + \frac{1}{4\pi}\int_{-1}^{1} \frac{d\gamma_1(\zeta)}{d\zeta} \operatorname{sgn}(z-\zeta)\log 2|z-\zeta| d\zeta \tag{5.25}$$

where E_0 is given in (5.24)

The leading term in the inner solution is $-\varepsilon A_0 \sin\alpha \log|x|$; this term is exactly matched to one term in the outer solution $-\frac{1}{2\pi}\gamma_1(z)\log|x|$. This shows that the anomaly in the classic solution is just part of a proper near-field expansion of the complete solution. It is also noted that far away from the planing surface, if AR approaches infinity or $\varepsilon \to 0$, physically it corresponds to a large span of small chord length and the planing surface shrinks to a line. To this lowest order, there is no disturbance in the far field, which agrees with the description in (5.25).

The location of the planing surface is now uniquely defined with respect to the undisturbed free surface. In natural variables, (5.25) gives

$$h = -\frac{a_0}{\pi(b_0+c_0)}[\sin\alpha \cdot \log\left(\frac{a_0}{\pi(b_0+c_0)}\right) - \pi(\cos\alpha - b_0) - M_0 \sin\alpha$$

$$- R_0\cos\alpha + \sin\alpha] + \frac{\sin\alpha}{2\pi}\int_{-1}^{1} \frac{d}{d\zeta}\left(\frac{a_0}{b_0+c_0}\right)\mathrm{sgn}\,(z-\zeta)\log 2|z-\zeta|d\zeta \tag{5.26}$$

where M_0, R_0 are defined in (4.13b). This is the important result of this chapter.

V. 5. Second-Order Inner Free-Surface Solution

Far away, the inversion formula gives:

$$\begin{aligned} Z = {} & (A_0 + A_1 + iA_0\frac{c_1}{\sin\alpha})\zeta + A_0(e^{-i\alpha}-b_0)\log\zeta \\ & + A_0e^{-i\alpha}(M_0-iR_0) + O(\tfrac{1}{\zeta}) \end{aligned} \tag{5.27}$$

On the free surface:

$$\begin{aligned} Z = X + iY = {} & (A_0+A_1+iA_0\frac{c_1}{\sin\alpha})\xi + A_0(e^{-i\alpha}-b_0)\log\xi \\ & + A_0e^{-i\alpha}(M_0-iR_0) + O(\tfrac{1}{\xi}) \quad \text{as } |\xi|\to\infty \end{aligned} \tag{5.28}$$

Separate the real and imaginary parts and eliminate the parameter ξ to give:

$$Y \sim \frac{c_1}{\sin\alpha}X - A_0\sin\alpha\log|X| + E_0 + O(\frac{\log|X|}{X}) \tag{5.29}$$

V. 6. Second Order Free-Surface Matching

Let m=2 and n=2 on (2.6):

2-T. I. E. of 2-T.O . E.

$$= \text{2-T. O. E. of 2-T. I. E.} \tag{5.30}$$

From (5.20), the two-term inner expansion of the two-term outer expansion is

$$\eta_0 + \eta_1 \sim -\frac{\gamma_1(z)}{2\pi}(1+\log|x|) + \frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1(\zeta)}{d\zeta}\,\text{sgn}\,(z-\zeta)$$
$$\cdot \log 2|z-\zeta|d\zeta - \frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1(\zeta)}{d\zeta}\,\frac{x}{z-\zeta}\,d\zeta \tag{5.31}$$

In (5.29), use outer variables to give the two-term outer expansion of the two-term inner expansion :

$$y \sim \eta_0 + \eta_1 \sim h + \frac{c_1}{\sin\alpha}x - \varepsilon A_0 \sin\alpha \log|x|$$
$$+ \varepsilon\log\varepsilon\, A_0 \sin\alpha + \varepsilon A_0[\sin\alpha\log A_0$$
$$- \pi(\cos\alpha - b_0) - M_0\sin\alpha - R_0\cos\alpha] \tag{5.32}$$

From (5.30), (5.31) and (5.32), matching show that each term in (5.31) is exactly matched by one term in (5.32), and vice versa. In a sense, this matching is trivial. However, it does give a hint that so far the analysis is correct.

VI. HYDRODYNAMIC-LIFT CALCULATION

VI. 1. Chord Length of the Planing Surface

The hydrodynamic problem can be completely described by parameters a_0, b_0, and c_0 for the first-order solution and a_1, b_1, and c_1 for the second-order solution respectively. However, those parameters are related to the chord length at each section. Let the chord length be $\ell(z)$ in natural variable and $L(z)$ in inner variable. The chord length is mapped onto the ζ-plane by the segment $|\xi| < 1$ on the real axis $\eta=0$ in Figure 3.2. The trailing edge and leading edge of the plate are mapped onto $\zeta= 1,-1$ respectively. From equation (3.11), the chord length is found to be

$$L(z)e^{-i\alpha} = Z_T - Z_L = \int_{-1}^{1} \frac{H(\zeta')}{W(Z(\zeta'))} d\zeta'$$

where Z_T and Z_L denote the trailing edge and leading edge locations, respectively. So

$$\ell(z) = \varepsilon e^{i\alpha} \int_{-1}^{1} \frac{H(\zeta')}{W(\zeta')} d\zeta' \tag{6.1}$$

A. First-Order Chord Length Expression

If (3.11) and (3.18) are substituted into (6.1), it follows that:

$$\ell(z) = \frac{\varepsilon\delta_0}{\pi(b_0+c_0)} [-(1+c_0b_0) \log \frac{b_0-1}{b_0+1} - 2c_0$$
$$+ \pi\sqrt{1-c_0^2}(b_0-\sqrt{b_0^2-1})] \tag{6.2}$$

The first-order parameters a_0, b_0, and c_0 are thus related to the chord length $\ell(z)$.

B. Second-Order Chord Length Expression

Substitute (4.32a) and (4.36) into (6.1) to give

$$\ell(z) = \frac{\varepsilon(\delta_0+\delta_1)}{\pi(b_0+b_1+c_0+c_1)}[- (1+(c_0+c_1)(b_0+b_1) \log \frac{b_0+b_1-1}{b_0+b_1+1}$$
$$-2(c_0+c_1) + \pi\sqrt{1-(c_0+c_1)^2}(b_0+b_1-\sqrt{(b_0+b_1)^2-1})] \tag{6.3}$$

Thus, the parameters a_1, b_1, and c_1 are also related to the chord length $\ell(z)$.

VI. 2. Wetted Length of the Planing Surface

A jet is thrown off at the leading edge. Therefore it is somewhat ambiguous to refer to the "wetted length" in this case. However, usually the jet thickness is small compared to the chord length. The "pressure length can thus be defined as the distance between the trailing edge and the stagnation point on the planing plate. Then define the "wetted length" $L_W(z)$ as the pressure length, *i.e.*,

$$L_W(z)e^{-i\alpha} = z_T - z_{Stag} = \int_{c^*}^{1} \frac{H(\zeta')}{W(\zeta')} d\zeta'$$

where z_{Stag} denotes the stagnation point.

To the first-order, it is

$$\ell_W(z) = \frac{\varepsilon\delta_0}{\pi(b_0+c_0)}\{(1+c_0b_0)\log\frac{b_0+1}{b_0+c_0} - c_0 + c_0^2$$
$$+ \sqrt{1-c_0^2}[\tfrac{\pi}{2}(b_0-\sqrt{b_0^2-1}) - \sqrt{1-c_0^2} + b_0\sin^{-1}(-c_0)$$
$$+ \sqrt{b_0^2-1}\sin^{-1}\frac{b_0c_0+1}{b_0+c_0}]\} \qquad (6.4)$$

To the second order, it is

$$\ell_W(z) = \frac{\varepsilon(\delta_0+\delta_1)}{\pi(b_0+b_1+c_0+c_1)}\{1+(c_0+c_1)(b_0+b_1)\log(\frac{b_0+b_1+1}{b_0+b_1+c_0+c_1})$$
$$- (c_0+c_1) + (c_0+c_1)^2 + \sqrt{1-(c_0+c_1)^2}[\tfrac{\pi}{2}(b_0+b_1-\sqrt{(b_0+b_1)^2-1})$$
$$- \sqrt{1-(c_0+c_1)^2} + (b_0+b_1)\sin^{-1}(-c_0-c_1)$$
$$+ \sqrt{(b_0+b_1)^2-1}.\sin^{-1}\frac{b_0c_0+1}{b_0+c_0}]\} \qquad (6.5)$$

VI. 3. Jet Direction

Let jet angle β be the angle the jet makes asymptotically with the plate. The jet J in the Z-plane is mapped onto $\zeta=-b_0$. Equation (3.18) gives

$$e^{(-\pi+\alpha-\beta)i} = e^{i\alpha}\frac{1+b_0c_0-i\sqrt{1-c_0^2}\sqrt{b_0^2-1}}{-b_0-c_0}$$

The real part gives

$$\beta = \cos^{-1}\left(\frac{1+c_0 b_0}{b_0+c_0}\right) \tag{6.6}$$

VI. 4. Lift Force

Lift force can be obtained by integrating pressure along the plate. This is a near-field problem.

Let P be the pressure in the fluid, F be the force and τ be the lift on the planing surface per-unit-length in spanwise direction. For a steady, irrotational, incompressible flow, the Bernoulli equation gives

$$\frac{P}{\rho} + \frac{1}{2}(\phi_x{}^2 + \phi_y{}^2 + \phi_z{}^2) = \frac{1}{2} + \frac{P_\infty}{\rho} \tag{6.7}$$

where ρ is the fluid density. Let $P_\infty=0$ and use inner variables to obtain:

$$\frac{P}{\rho} + \frac{1}{2}(\phi_X{}^2 + \phi_Y{}^2 + \varepsilon^2\phi_Z{}^2) = \frac{1}{2} \tag{6.8}$$

Now assume the following asymptotic expansions exist for P,F, and τ

$P \sim \sum_{n=0}^{N} P_n$ $\qquad P_{n+1}=o(P_n)$ as $\varepsilon\to 0$ for fixed X,Y

$F \sim \sum_{n=0}^{N} F_n$ $\qquad F_{n+1}=o(F_n)$ as $\varepsilon\to 0$ for fixed X,Y

$$\tau \sim \sum_{n=0}^{N} \tau_n \qquad \tau_{n+1} = o(\tau_n) \quad \text{for fixed } X,Y \tag{6.9}$$

Substitute (6.9) into (6.8) to give

$$\frac{P_0}{\rho} + \frac{1}{2}(\Phi_{0X}^{\ 2} + \Phi_{0Y}^{\ 2}) + \frac{P_1}{\rho} + \Phi_{0X}\Phi_{1X} + \Phi_{0Y}\Phi_{1Y}$$
$$+ \frac{P_2}{\rho} + \frac{1}{2}(\Phi_{1X}^{\ 2} + 2\Phi_{0X}\Phi_{2X} + \Phi_{1Y}^{\ 2} + 2\Phi_{0Y}\Phi_{1Y} + \varepsilon^2\Phi_{0Z}^{\ 2})$$
$$+ \ldots\ldots = \frac{1}{2} \tag{6.10}$$

A. First-Order Lift Force

Keeping only the leading-order terms in (6.10), one obtains:

$$\frac{P_0}{\rho} + \frac{1}{2}(\Phi_{0X}^{\ 2} + \Phi_{0Y}^{\ 2}) = \frac{1}{2} \tag{6.11}$$

The force on the planing plate is

$$F_0 = \int_{\ell.e.}^{t.e} P_0 dZ \tag{6.12}$$

where ℓ.e. and t.e. denote the leading edge and trailing edge of the planing plate respectively. The integral is taken along the planing plate. This force is normal to the plate and can be decomposed into vertical and horizontal components, that is, lift and drag, respectively. The lift force per-unit-length can now be obtained from equation (6.12) to give:

$$\tau_0(z) = Re(F_0) = Re\ \{\int_{\ell.e.}^{t.e.} \frac{\rho}{2}(1-|W_0|^2)dZ\} \tag{6.13}$$

If equations (3.16) and (3.10) are substituted into equation (6.13), one obtains:

$$\tau_0(z) = \rho\frac{a_0}{b_0+c_0}\ (b_0-\sqrt{b_0^2-1})\sqrt{1-c_0^2}\cos\alpha \tag{6.14}$$

where a_0, b_0, and c_0 can be obtained by solving three simultaneous equations (3.31), (5.26) and (6.2). The equation (6.14) gives the lift force per-unit-length in spanwise direction. This equation agrees with Green's non-linear solution. However, Green could only compute the lift force in two special cases. One case is $b_0 \to 1$, which corresponds to the infinitely long planing plate, that is, the very lightly loaded planing plate. The other case is $b_0 \to \infty$, corresponding to Rayleigh's cavity flow which is of no interest in the planing problem. In the present work, the lift force can be computed uniquely for arbitrary angle of incidence and for a heavily-loaded as well as a lightly-loaded planing surface. Numerical calculations will be presented in the next section.

B. Second-Order Lift Force

Equation (6.10) gives

$$\frac{P_0+P_1}{\rho} + \frac{1}{2}[(\Phi_0+\Phi_1)_X^2 + (\Phi_0+\Phi_1)_Y^2] = \frac{1}{2} \tag{6.15}$$

The force per-unit-length on the planing plate to the second-order is:

$$F_0 + F_1 = \int_{\ell.e.}^{t.e.} (P_0+P_1)dZ \tag{6.16}$$

If equations (4.32), (4.36) and (6.15) are substituted into equation (6.16), the lift force to the second-order is found to be

$$\tau_0(z) + \tau_1(z) = \rho\frac{a_0+a_1}{(b_0+b_1+c_0+c_1)}\sqrt{1-(c_0+c_1)^2}$$
$$(b_0+b_1-\sqrt{(b_0+b_1)^2-1})\cos\alpha \tag{6.17}$$

VI. 5. First-Order Numerical Calculation

There are three unknowns and three available equations (3.31), (5.26), and (6.2). Therefore, it appears likely that a solution can be found. It is noted that c_0 is function of α only and is easily obtained. The problem can thus be reduced to two unknowns with two equations as follow:

$$h = -\frac{a_0}{\pi(b_0+c_0)}[\sin\alpha\cdot\log\frac{a_0}{\pi(b_0+c_0)} - \pi(\cos\alpha-b_0)$$
$$- M_0\sin\alpha - R_0\cos\alpha + \sin\alpha] + \frac{\sin\alpha}{2\pi}\int_{-1}^{1}[\frac{d}{d\zeta}(\frac{a_0}{b_0+c_0})$$
$$\cdot\mathrm{sgn}(z-\zeta)\cdot\log 2|z-\zeta|d\zeta \tag{6.18}$$

$$\ell(z) = \frac{a_0}{\pi(b_0+c_0)}[-(1+c_0b_0)\cdot\log\frac{b_0-1}{b_0+1} - 2c_0$$
$$+ \pi\sqrt{1-c_0^2}(b_0-\sqrt{b_0^2-1})] \tag{6.19}$$

Now a_0 can be expressed as a function of b_0, from (6.19). If a_0, so obtained, is substituted back into (6.18), it gives a non-linear integral equation of one variable b_0 with a complicated kernel. To perform a numerical calculation, it is more convenient to solve a system of two non-linear equations (6.18) and (6.19) simultaneously than a non-linear integral equation with a very complicated kernel.

Now, express (6.18) and (6.19) symbolically as following:

$$h = g(a_0^{(n)}, b_0^{(n)}) + \text{I.P.}(a_0^{(n-1)}, b_0^{(n-1)})$$
$$\ell(z) = f(a_0^{(n)}, b_0^{(n)}) \qquad (6.20)$$

Where I.P. represents the integral term in (6.18), and the superscript $^{(n)}$ denotes a value at the n-th iteration. For the first iteration, integral part is set equal to zero. The problem requires the solution of the two non-linear algebraic equations. Then the iteration is terminated if

$$\left| \frac{a_0^{(n)} - a_0^{(n-1)}}{a_0^{(n)}} \right| < \delta$$

and

$$\left| \frac{b_0^{(n)} - b_0^{(n-1)}}{b_0^{(n)}} \right| < \delta \qquad (6.21)$$

where δ is some positive number which is much less than one.

A. Cusp Shape

Let the chord-length distribution be

$$\ell(z) = \ell_0(1-z^2)^{1.5} \tag{6.22}$$

where ℓ_0 is the chord length at middle section, z=0.

The chord length $\ell(z)$ and its first derivative with respect to z are continuous at the tip, as elsewhere. Recall that h is the height of the trailing edge above the undisturbed free surface; physically, it can be related to a loading condition on the planing surface. If the loading is heavy, the trailing edge will be located much below the undisturbed free surface and consequently h will have a large negative value.

For $\ell_0=0.4$, $\alpha=15^0$, the coefficient of lift-per-unit span (local lift coefficient) formed with the wetted length is given in Figure 6.1. for different value of h. Note that the ordinate scale does not start at zero. The jet thickness a_0 at each section is shown in Figure 6.2.

A cusp shape is not a practical shape. More detail will be discussed for the elliptic shape.

B. Elliptic Shape

Let the chord-length distribution be

$$\ell(z) = \ell_0(1-z^2)^{0.5} \tag{6.23}$$

The chord length is continuous at the tip, as elsewhere; however its first derivative is not continous at the tip.

$$\ell = 0.4(1-z^2)^{1.5}$$
$$\alpha = 15^0$$

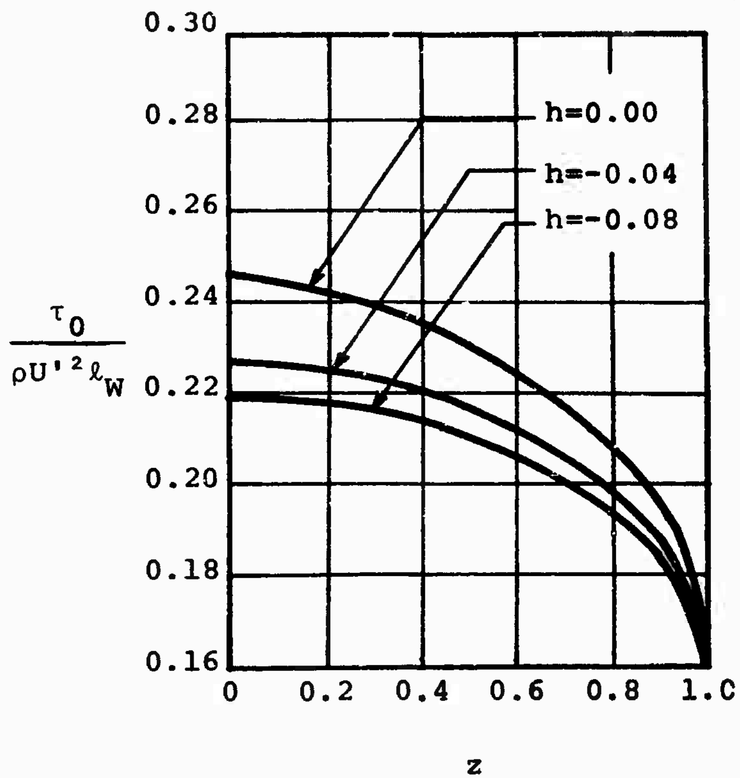

Figure 6.1 Local Lift Coefficient

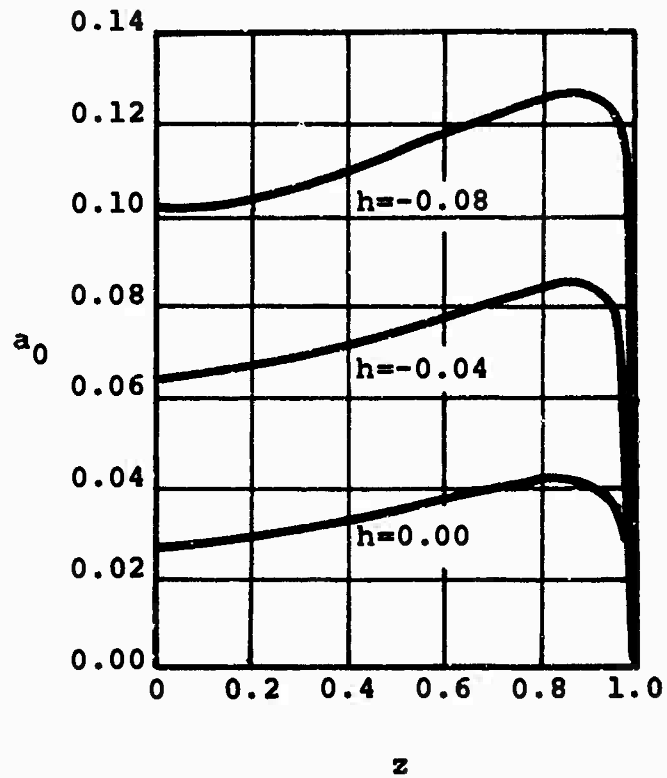

Figure 6.2 Jet Thickness

Observation from experiments indicates that tip effects are local phenomena, and the solution obtained from the present theory can be expected to give a good result except in the very neighborhood of the tips.

Local lift coefficient distribution in spanwise direction for $\alpha=15^0$, 10^0, and 5^0 are plotted in Figures 6.3, 6.4, and 6.5 respectively for the case $\ell_0=0.4$. From the definition of b_0, the value of b_0 is bounded by

$$1 < b_0 < \infty$$

For the case $b_0 \to 1$, if equations (6.2) and (6.4) are used, one obtains:

$$\frac{\ell_W}{\ell_c} \sim 0 \qquad \text{as } b_0 \to 1$$

where ℓ_c is the chord length in natural variable. From equations (6.4) and (6.14), numerical calculation gives:

$$\frac{\tau_0}{\rho U^2 \ell_W} \sim \begin{matrix} 0.289 \\ 0.214 \\ 0.120 \end{matrix} \qquad \text{for} \quad \begin{matrix} \alpha=15^0 \\ \alpha=10^0 \\ \alpha=5^0 \end{matrix} \tag{6.23}$$

The upper-bound limit for the local-lift-coefficient is thus obtained. Note that this upper-bound limit is a function of α only. This statement can also be seen in

in Figure 6.8.

For the case $b_0 \to \infty$, if equations (6.2) and (6.4) are used, one obtains:

$$\frac{\ell_W}{\ell_c} \sim 1 \qquad \text{as } b_0 \to \infty$$

Now, if equation (6.6) is used, the jet angle β is found to be:

$$\beta \sim 180^0 \qquad \text{as } b_0 \to \infty$$

This is a cavity-flow. However, this does provide a lower-bound for the extremely heavily loaded planing surface. Numerical calculation gives:

$$\frac{\tau_0}{\rho U^2 \ell_W} \sim \begin{array}{ll} 0.167 & \text{for } \alpha=15^0 \\ 0.121 & \alpha=10^0 \\ 0.066 & \alpha=5^0 \end{array} \qquad (6.24)$$

If equation (6.18) is used, the depth of submergence h is found to be

$$h \sim -\infty \qquad \text{as } b_0 \to \infty$$

Those upper-bound and lower-bound limits are plotted in Figures 6.3, 6.4, and 6.5 respectively. Note that due

$\ell = 0.4(1-z^2)^{0.5}$

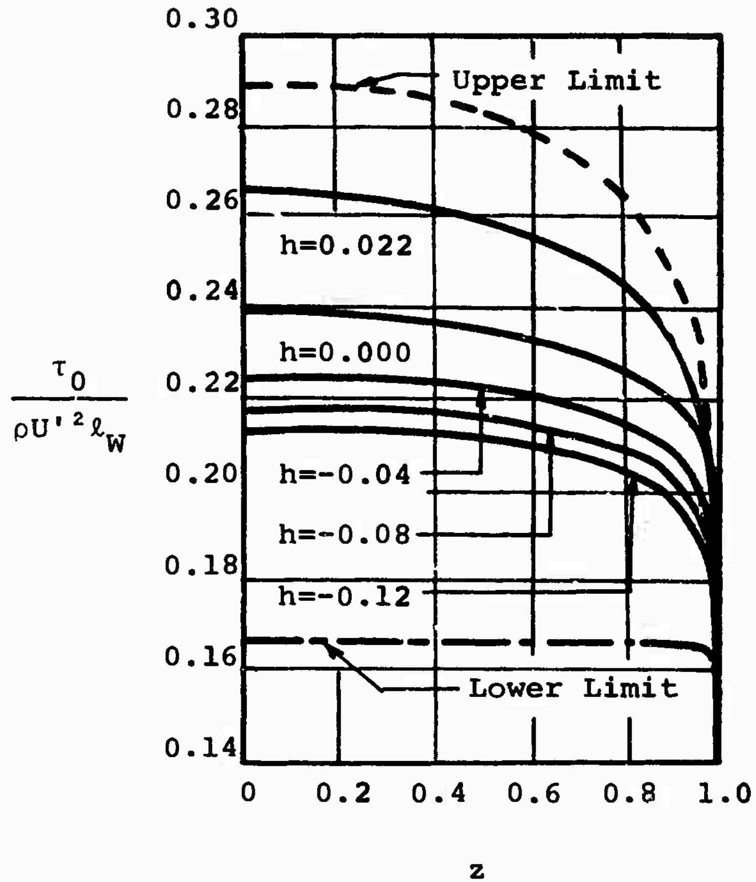

Figure 6.3 Local Lift Coefficient

$\alpha = 15^0$

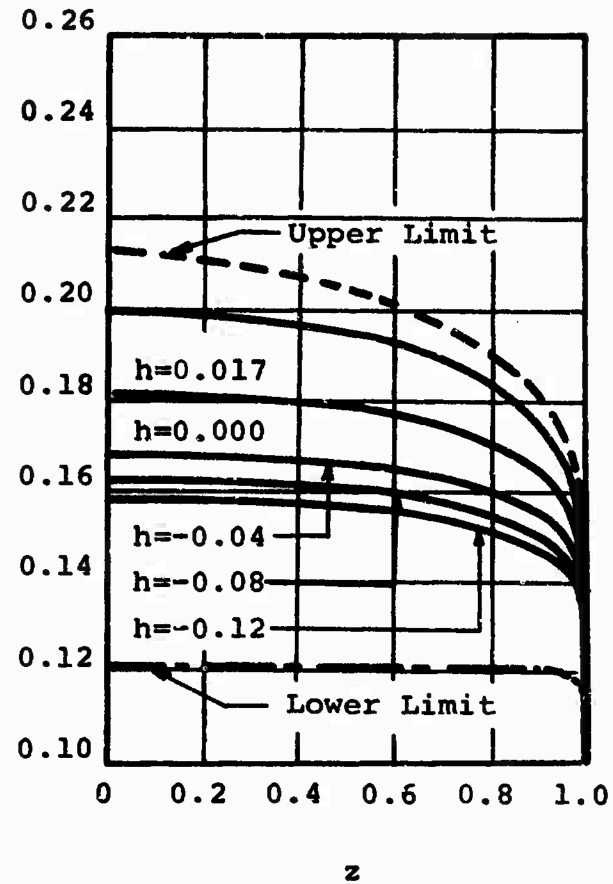

Figure 6.4 Local Lift Coefficient

$\alpha = 10^0$

$\ell = 0.4(1-z^2)^{0.5}$

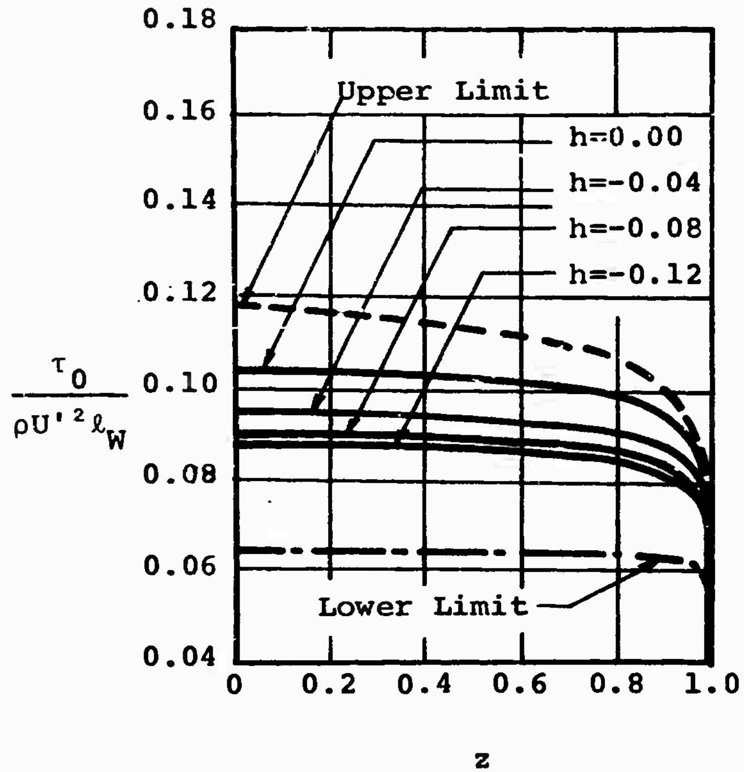

Figure 6.5 Local Lift Coefficient
$\alpha = 5^0$

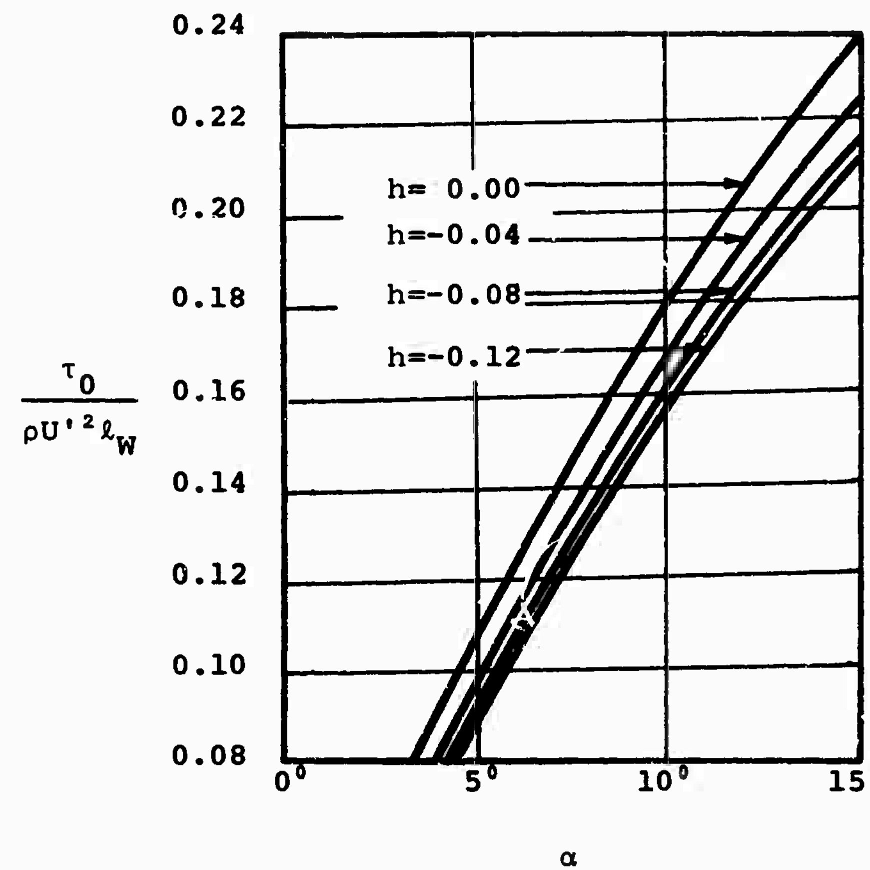

Figure 6.6 Local Lift Coefficient
(z = 0)

$$\ell = \ell_0 (1-z^2)^{0.5}$$

for $\alpha = 1^{0}$ at the middle section

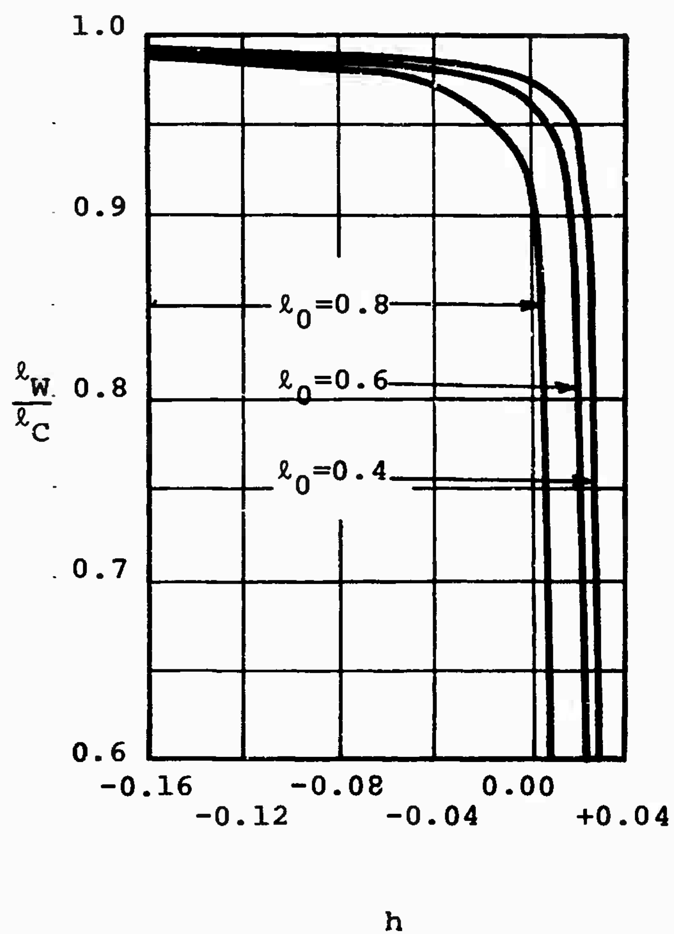

Figure 6.7 $\frac{\ell_W}{\ell_c}$ vs. h

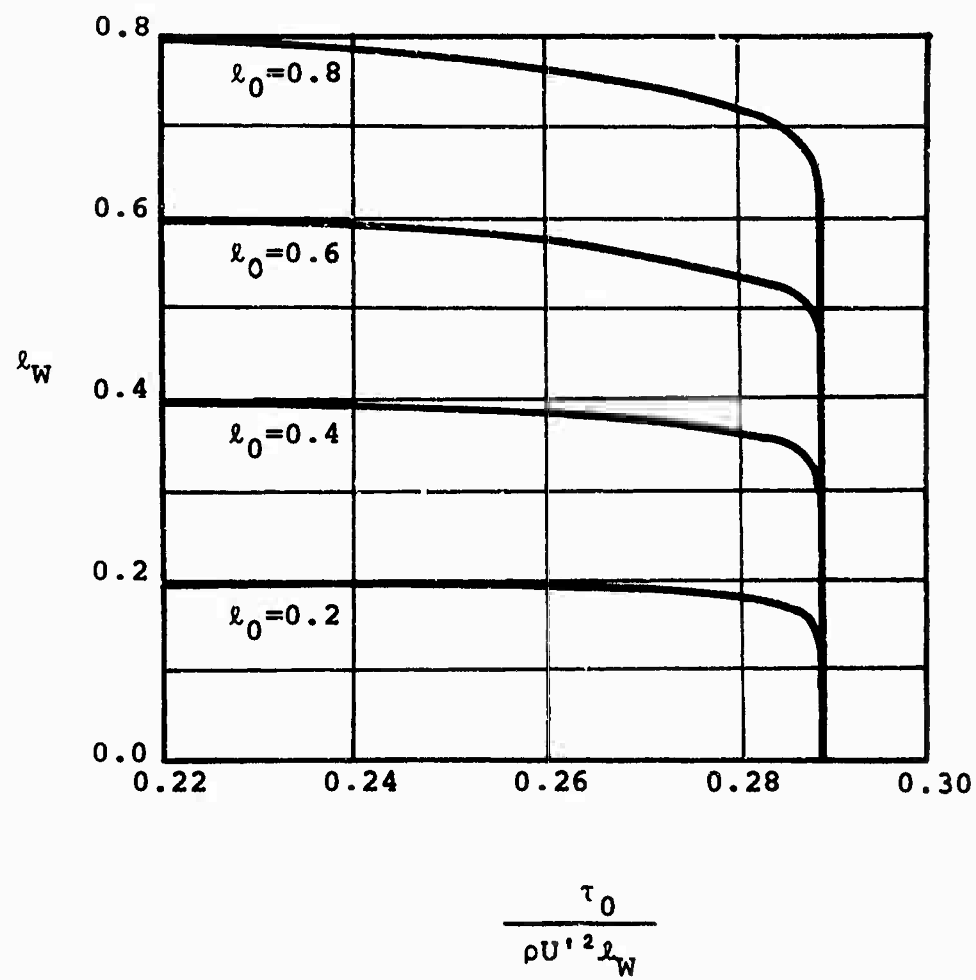

Figure 6.8 Local Lift Coefficient vs. ℓ_W

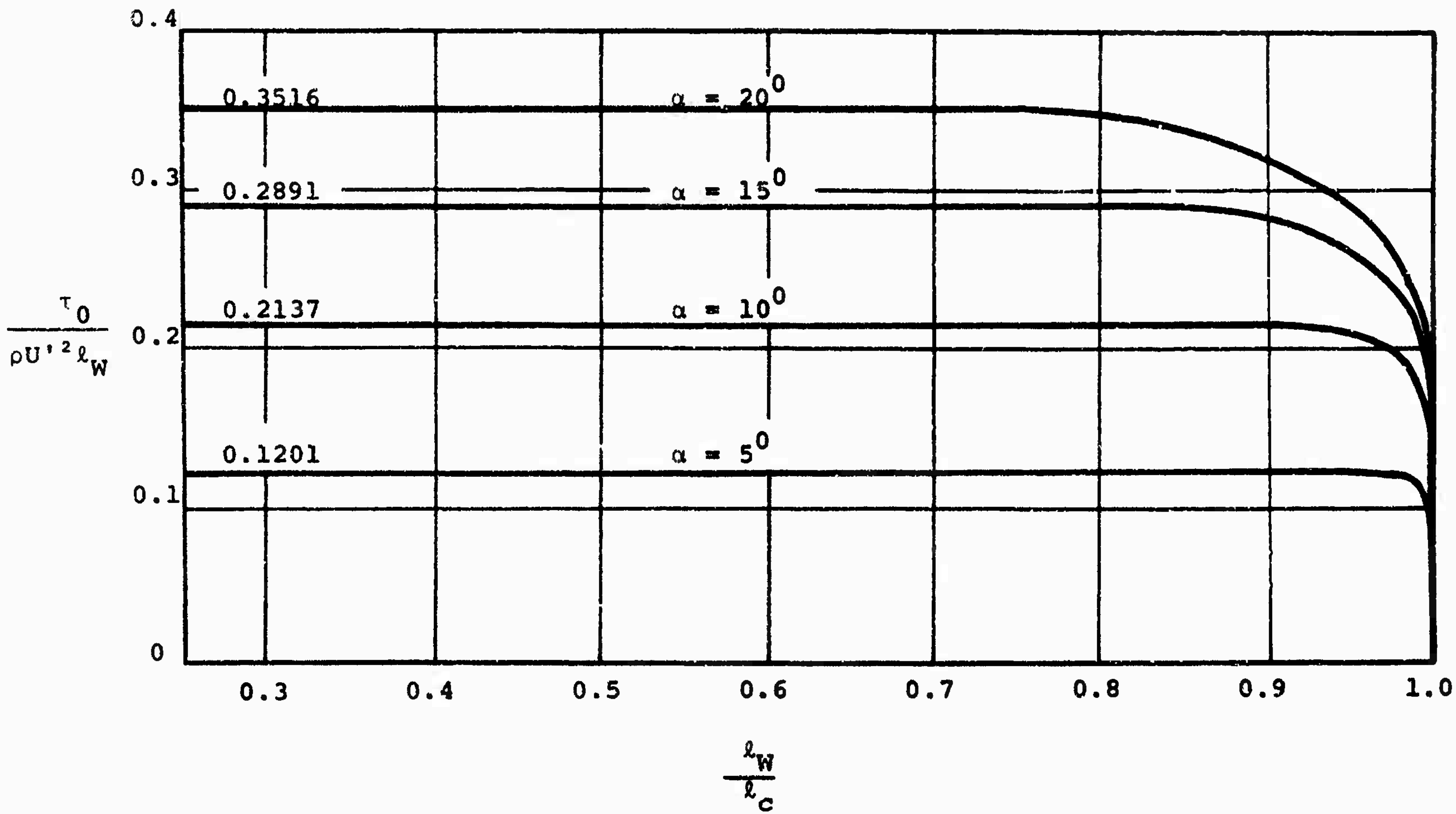

Figure 6.9 Local Lift Coefficient vs. $\frac{\ell_W}{\ell_c}$

to the three-dimensional effect, the values of upper-bound and lower-bound local lift coefficients decrease along the spanwise direction and approach zero at the tips.

The local lift coefficient at the middle section is plotted with respect to angle of attack α in Figure 6.6.

In Figure 6.7, the ratio of wetted length to chord length $R_\ell = \frac{\ell_W}{\ell_c}$, at the middle section is plotted against the height of the trailing edge for different values of ℓ_0.

In Figure 6.8, ℓ_W, at the middle section, is plotted against local lift coefficient for different values of ℓ_0 for $\alpha=15^0$. It shows that for a small value of ℓ_W, say, $\ell_W<0.15$, the chord length has no significance. The hydrodynamic results are the same whether ℓ_0 is 0.8, 0.6, 0.4, or 0.2.

In Figure 6.9, R_ℓ is plotted against the local lift coefficient. This figure can also be obtained from Green's non-linear solution. It is seen that the local lift coefficient takes a constant value for a fixed α when R_ℓ is small. For the case $\alpha\to 0$, from equations (6.4) and (6.14), one obtains:

$$\frac{\tau_0}{\rho U^2 \ell_W} \sim \frac{1}{2}\pi\alpha \qquad \text{as } \alpha\to 0 \qquad (6.25)$$

Wagner (1932) considered the linearized case of an infinitely long plate and very small α. Wagner gave the

result as given in equation (6.25). Therefore, the present non-linear theory does approach Wagner's linearized solution as $\alpha \to 0$. For a finite value of α, the present theory gives the values of local lift coefficient in equation (6.23), while Wagner's linearized theory gives the value of 0.411, 0.274, and 0.137 for $\alpha = 15^0$, 10^0, and 5^0 respectively. It is noted that a non-linear effect must be considered if α is not very small.

VI. 6. Second-Order Numerical Calculation

In order to obtain a unique second-order solution, the parameters a_1, b_1, and c_1 must be determined. There are three unknowns, but so far only two equations (4.45) and (6.2b), have been obtained. One is obtained from the velocity potential matching, and the other is from the chord-length relationship. The second-order free-surface matching fails to provide an additional equation to define the depth of submergence.

From the first-order solution, it appeared that a second-order outer solution had to be found to provide a height reference for the first-order complete solution. Accordingly, a third-order outer solution must be found in order to provide a height reference for the second-order complete solution. An equation similar to (5.26) will then be obtained from the free surface-matching. Then the

second-order hydrodynamic problem will be completely solved. The third-order outer solution will be formulated in the next chapter. However, at the present stage, two special cases can be readily calculated before entering the complicated third-order outer solution.

A. $b^* = b_o + b_1 \to 1$

This corresponds to a lightly loaded planing surfaces. The solution similar to Wagner's, with a second-order correction, can be obtained.

B. Angle of Attack α Is Small

The parameter c_1 which represents a downwash correction, is given in (4.43). It is a function of a_0, b_0, c_0, and α only and can be obtained easily from the first-order solution. Now, if the angle of incidence is small, the problem can be linearized as in aerodynamics, so that the second-order effect is mainly due to downwash. One obtains:

$$\tau_0(z) + \tau_1(z) \sim \rho \frac{a_0}{(b_0+c_0+c_1)} \sqrt{1-(c_0+c_1)^2}\,(b_0-\sqrt{b_0^{\,2}-1})\cos\alpha \tag{6.26}$$

The solution of c_1 is given in Figure 6.10 for the case of an elliptical planing surface. As expected, the solution for c_1 breaks down at the tips. However, the curves indicate that it is quite a weak singularity at the tips. The solution can be expected to be valid all across the span

except in the very neighborhood of the tips. Local lift coefficient is given in Figure 6.11. For a heavily loaded planing surface, the downwash correction is more significant than a lightly loaded planing surface.

$\ell = 0.4(1-z^2)^{0.5}$
$\alpha = 15^0$

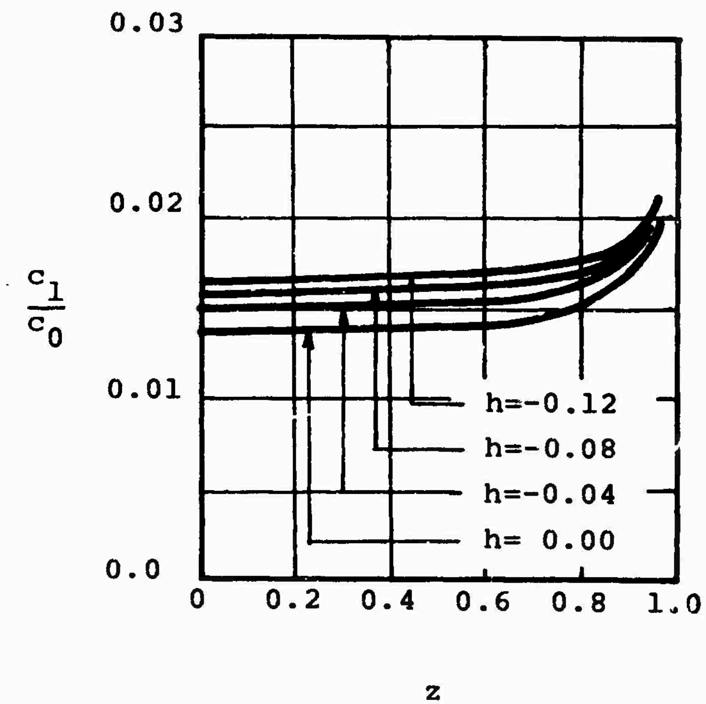

Figure 6.10 Downwash Distribution Along the Span

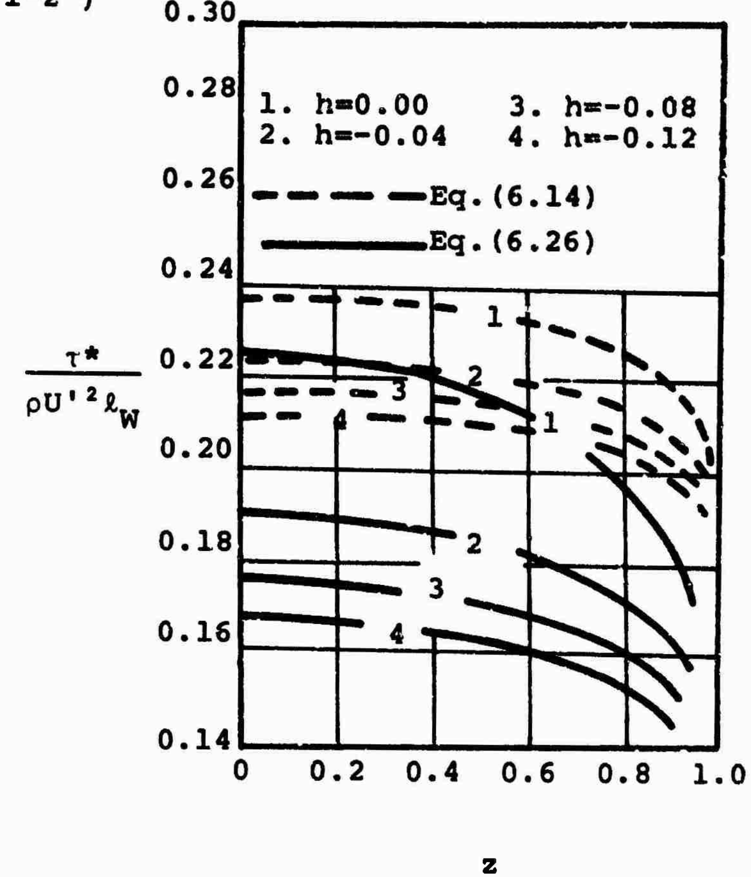

Figure 6.11 Downwash Correction

VII. THIRD-ORDER OUTER SOLUTION

VII. 1. Introduction

In order to obtain a complete second-order solution, it is necessary to solve the far-field problem to three terms. A third-order free-surface expression relating parameters a_1, b_1, and c_1 to the depth of submergence similar to (5.26), can then be obtained from the matching of the three-term outer expansion of the two-term inner expansion to the two-term inner expansion of the the three-term outer expansion. The second-order problem can thus be completely solved.

VII. 2. Third-Order Outer Boundary-Value Problem

From (2.13) - (2.16), the following are obtained:

$$\text{(L)} \qquad \phi_{2xx} + \phi_{2yy} + \phi_{2zz} = 0 \quad \text{in fluid} \tag{7.1}$$

$$\text{(F1)} \qquad \phi_{1x}^{2} + 2\phi_{2x} + \phi_{1y}^{2} + \phi_{1z}^{2} + 2\,\eta_1\phi_{1xy} = 0 \quad \text{on } y=0 \tag{7.2}$$

$$\text{(F2)} \qquad \phi_{1x}\eta_{1x} + \eta_{2x} - \phi_{2y} + \phi_{1z}\eta_{1z} - \eta_1\phi_{1yy} = 0 \quad \text{on } y=0 \tag{7.3}$$

$$\text{(R)} \qquad \phi_{2x} = 0 \;;\; \phi_{2y} = \phi_{2z} = 0 \quad \text{as } x\to-\infty, y\to-\infty \tag{7.4}$$

The free-surface boundary condition (F1) can be rewritten as:

$$\phi_{2x} = -\frac{1}{2}(\phi_{1x}^{2}+\phi_{1y}^{2}+\phi_{1z}^{2}+2\eta_1\phi_{1xy}) \quad \text{on } y=0 \qquad (7.5)$$

The right-hand side of the above equation is known from the first-and second-order outer solutions. Therefore, combined with Laplace equation (7.1), the problem is a non-homogeneous boundary-value problem. The governing differential equation is linear and so the principle of superposition is applicable. The final solution can be obtained by adding the homegeneous solution to the particular solution.

Define the velocity potential ϕ_{2p} to be a particular solution of this non-homogeneous problem. And let p^* be

$$p^* = -\frac{\rho}{2}(\phi_{1x}^{2}+\phi_{1y}^{2}+\phi_{1z}^{2}+2\eta_1\phi_{1xy}) \quad \text{on } y=0 \qquad (7.6)$$

Then the dynamic condition (F1) becomes

$$\phi_{2x} = \frac{p^*}{\rho} \qquad (7.7)$$

The equation (7.7) can be interpreted in terms of a pressure distribution over the free surface. This problem will be solved by the use of a double Fourier transform. (See Wehausen & Laitone[31])

VII. 3. Third-Order Outer Solution

Define the double Fourier transform with respect to x and z to be

$$\tilde{\phi}(k,y,\theta) = \frac{k}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y,z)e^{-ik(x\cos\theta+z\sin\theta)}dx\cdot dz \qquad (7.8a)$$

Then the inverse transform is

$$\phi(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\phi}(k,y,\theta)e^{ik(x\cos\theta+z\sin\theta)}d\theta\cdot dk \qquad (7.8b)$$

Now take the double Fourier transform of the dynamic condition (F1) to give:

$$\tilde{\phi}_2 = \frac{\sec\theta}{ik\rho}\cdot\tilde{p}^* \qquad \text{on } y=0 \qquad (7.9)$$

Next, take the transform of the Laplace equation to give

$$\tilde{\phi}_{2yy} - k^2\tilde{\phi}_2 = 0$$

This is an ordinary differential equation. A general solution is

$$\tilde{\phi}_{2p}(k,y,\theta) = A(k,y,\theta)e^{-ky} + B(k,y,\theta)e^{ky} \qquad (7.10)$$

Note that y vertically up is positive. If radiation condition is used, equation (7.10) becomes:

$$\tilde{\phi}_{2p}(k,y,\theta) = B(k,y,\theta)e^{ky} \tag{7.11}$$

From (7.9) the constant B is determined to give

$$B(k,y,\theta) = \frac{\sec\theta}{ik\rho}\tilde{p}^{*} \tag{7.12}$$

So the equation (7.11) gives

$$\tilde{\phi}_{2p}(k,y,\theta) = e^{ky}\cdot\frac{\sec\theta}{i\rho 2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p^{*}(x,z)e^{-ik(x\cos\theta+z\sin\theta)}dx\cdot dz \tag{7.13}$$

Take the inverse transform to give

$$\begin{aligned}\phi_{2p}(x,y,z) = \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\pi}^{\pi} \{&\frac{e^{ky}\sec\theta}{i2\pi\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p^{*}(m,y,n) \\ &\cdot e^{-ik(m\cos\theta+n\sin\theta)}dm\cdot dn\} \\ &\cdot e^{ik(x\cos\theta+z\sin\theta)}dk\cdot d\theta\end{aligned}$$

Integration with respect to k gives:

$$\begin{aligned}\phi_{2p}(x,y,z) = -\frac{1}{4\pi^2}\int_{-\pi/2}^{\pi/2}\sec\theta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}&(\phi_{1m}^{2}+\phi_{1y}^{2}+\phi_{1n}^{2}+2\phi_{1my}\eta_1) \\ &\cdot\frac{(x-m)\cos\theta+(z-n)\sin\theta}{y^2+[(x-m)\cos\theta+(z-n)\sin\theta]^2}d\theta\cdot dm\cdot dn\end{aligned} \tag{7.14}$$

where $\phi_1(x,y,z)$ and $\eta_1(x,z)$ are given in (4.11) and (5.17). The strength $\gamma_1(z)$, $\mu_1(z)$, and $\lambda_1(z)$ are given in (4.19).

Equation (7.14) gives a particular solution. The homogeneous solution is obtained from (4.8); it gives:

$$\phi_{2h}(x,y,z) = \frac{1}{4\pi}\int_{-1}^{1} \frac{y\gamma_2(\zeta)}{y^2+(z-\zeta)^2}[1+\frac{x}{\sqrt{x^2+y^2+(z-\zeta)^2}}]d\zeta$$
$$+ \frac{1}{4\pi}\int_{-1}^{1} \frac{y\mu_2(\zeta)}{[x^2+y^2+(z-\zeta)^2]^{3/2}}\cdot d\zeta$$
$$+ \frac{1}{4\pi}\int_{-1}^{1} \frac{x\lambda_2(\zeta)}{[x^2+y^2+(z-\zeta)^2]^{3/2}}\cdot d\zeta \tag{7.15}$$

The first integral in the above equation represents the correction $\gamma_2(z)$ to the vortex strength $\gamma_1(z)$ in (4.9).

The final third-order outer solution is

$$\phi_2(x,y,z) = \phi_{2p}(x,y,z) + \phi_{2h}(x,y,z) \tag{7.16}$$

The unknowns $\gamma_2(z)$, $\mu_2(z)$, and $\lambda_2(z)$ can be determined by intermediate matching to the second-order inner solution.

VII. 4. Intermediate Velocity-Potential Matching

Let m=2 and n=3 in the equation (2.6):

2-T. I. E. of 3-T. O. E.

= 3-T. O. E. of 2-T. I. E. (7.18)

The three-term outer expansion is

$$\phi_0+\phi_1+\phi_2 \sim x + \frac{y}{4\pi}\int_{-1}^{1} \frac{\gamma_1(\zeta)}{y^2+(z-\zeta)^2}\left[1+\frac{x}{\sqrt{x^2+y^2+(z-\zeta)^2}}\right]d\zeta$$
$$+ \frac{y}{4\pi}\int_{-1}^{1} \frac{\gamma_2(\zeta)}{y^2+(z-\zeta)^2}\left[1+\frac{x}{\sqrt{x^2+y^2+(z-\zeta)^2}}\right]d\zeta$$
$$+ \frac{y}{4\pi}\int_{-1}^{1} \frac{\mu_2(\zeta)}{[x^2+y^2+(z-\zeta)^2]^{3/2}}\, d\zeta$$
$$+ \frac{x}{4\pi}\int_{-1}^{1} \frac{\lambda_2(\zeta)}{[x^2+y^2+(z-\zeta)^2]^{3/2}}\, d\zeta$$
$$- \frac{1}{4\pi^2}\int_{-\pi/2}^{\pi/2} \sec\theta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\phi_{1m}{}^2+\phi_{1y}{}^2+\phi_{1n}{}^2+2\phi_{1my}\eta_1)$$
$$\cdot \frac{(x-m)\cos\theta+(z-n)\sin\theta}{y^2+[(x-m)\cos\theta+(z-n)\sin\theta]^2}\, d\theta\cdot dm\cdot dn \qquad (7.19)$$

Take inner variables, the two-term inner expansion of the three-term outer expansion in outer variable is found to be:

$$\phi_0+\phi_1+\phi_2 \sim x - \frac{\gamma_1(z)}{2\pi}\tan^{-1}\frac{y}{x} - \frac{\gamma_2(z)}{2\pi}\tan^{-1}\frac{y}{x}$$
$$+ \frac{1}{2\pi}\frac{y\cdot\mu_2(z)}{x^2+y^2} + \frac{1}{2\pi}\frac{x\cdot\lambda_2(z)}{x^2+y^2}$$
$$- \frac{1}{4\pi^2}\int_{-\pi/2}^{\pi/2} \sec\theta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\phi_{1m}{}^2+\phi_{1y}{}^2+\phi_{1n}{}^2+2\phi_{1my}\eta_1)$$
$$\cdot \frac{d\theta\cdot dm\cdot dn}{-m\cos\theta+(z-n)\sin\theta} \qquad (7.20)$$

Next, consider the inner solution. After a long and tedious algebraic manipulation, the mapping inversion formula gives Z in terms of large ζ to three-term:

$$
\begin{aligned}
z \sim{} & \zeta\{A_0+A_1+A_2-A_0\frac{1+c_0{}^2}{2\sin^2\alpha}\cdot c_1{}^2 + i[A_0\frac{c_1}{\sin\alpha} + A_1\frac{c_1}{\sin\alpha} \\
& + A_0\cdot\frac{c_0(1+c_0{}^2)}{2\sin^3\alpha}\cdot c_1{}^2]\} + \{A_0[-c_0-b_0-(b_0c_1\cot\alpha \\
& + b_1\sin\alpha)\sin\alpha - (c_0b_1+c_1b_0)c_0] - A_1(c_0+b_0) \\
& - i[A_0\sin\alpha - A_0c_0(b_0c_1\cot\alpha+b_1\sin\alpha) \\
& + A_0\sin\alpha(c_0b_1+c_1b_0) + A_1\sin\alpha]\}\log\zeta \\
& + e^{-i\alpha}[A_0(M_0-iR_0) + A_0(M_1+iR_1) + A_1(M_0-iR_0)] \\
& + \frac{A_0}{\zeta}\{-c_0b_0(1+c_0b_0) + T_0\sin^2\alpha - i\sin\alpha[(1+c_0b_0)b_0 \\
& + c_0T_0]\}+O(\frac{1}{\zeta^2})
\end{aligned} \tag{7.21}
$$

where

$$
\begin{aligned}
T_0 &= \frac{b_0-2}{4} - \frac{\sqrt{b_0{}^2-1}}{b_0-\sqrt{b_0{}^2-1}} + b_0\sqrt{b_0{}^2-1} \\
A_1 &= A_0(\frac{\delta_1}{\delta_0} - \frac{b_1+c_1}{b_0+c_0}) \\
A_2 &= A_0(b_1+c_1+\frac{\delta_1}{\delta_0})\frac{b_1+c_1}{b_0+c_0} \\
M_1 &= c_1 - (1+c_0b_0)\frac{b_1}{b_0+1} - (c_0b_1+c_1b_0)\log(b_0+1) \\
R_1 &= (\frac{c_0b_0c_1}{\sin\alpha} - b_1\sin\alpha)\log 2 + b_1\sin\alpha \\
&\quad +(\frac{\sqrt{b_0{}^2-1}}{\sin\alpha}c_0c_1 - b_0b_1\sin\alpha)\log(b_0-\sqrt{b_0{}^2-1})
\end{aligned} \tag{7.22}
$$

Next, express ζ in terms of large z to three-term:

$$\zeta \sim \frac{Z}{A_0}\{1 - \frac{A_1}{A_0} - \frac{A_2}{A_0} - \frac{c_1^{\,2}}{2} + (\frac{A_1}{A_0})^2$$

$$- i[\frac{c_1}{\sin\alpha}(1-\frac{A_1}{A_0}) + \frac{c_0(1+c_0^{\,2})}{2\sin^3\alpha}c_1^{\,2}]\}$$

$$- \{(e^{-i\alpha}-b_0) - c_0(c_0b_1+c_1b_0) - (b_0c_1\cot\alpha+b_1\sin\alpha)\sin\alpha$$

$$- \frac{A_1}{A_0}(c_0+b_0) - i[c_0b_1+c_1b_0)\sin\alpha - (b_0c_1\cot\alpha$$

$$+ b_1\sin\alpha)c_0 + \frac{A_1}{A_0}\sin\alpha] - (\frac{A_1}{A_0} + i\frac{c_1}{\sin\alpha})(e^{-i\alpha}-b_0)\}$$

$$\cdot(\log Z - \log A_0) - e^{-i\alpha}(M_0-iR_0)$$

$$+(\frac{A_1}{A_0} + i\frac{c_1}{\sin\alpha})(e^{-i\alpha}-b_0) - e^{-i\alpha}[(\frac{A_1}{A_0} +i\frac{c_1}{\sin\alpha})(M_0-iR_0)$$

$$+ M_1 + \frac{A_1}{A_0}M_0 + i(R_1-\frac{A_1}{A_0}R_0)]$$

$$+ \frac{A_0}{Z}(e^{-i\alpha}-b_0)^2\cdot(\log Z-\log A_0)$$

$$+ \frac{A_0}{Z}[e^{-i\alpha}(M_0-iR_0)(e^{-i\alpha}-b_0) + c_0b_0(1+c_0b_0)$$

$$- T_0\sin^2\alpha + ic_0T_0\sin\alpha+(1+c_0b_0)b_0] + O(\frac{1}{Z^2}) \qquad (7.23)$$

If equation (7.23) is used, the second-order inner velocity potential expanded with respect to large Z to three-term can be obtained. Take outer variables, one obtains:

$$\varepsilon(\Phi_0+\Phi_1) \sim x + y\frac{c_1}{\sin\alpha} - \varepsilon(A_0+A_1)\sin\alpha\cdot\tan^{-1}\frac{y}{x}$$

$$- \frac{c_1^{\,2}}{2}x + \frac{yc_0(1+c_0^{\,2})}{2\sin^3\alpha} + \varepsilon A_0[(b_0c_1\cot\alpha$$

$$+ b_1\sin\alpha)c_0 - (c_0b_1+c_1b_0)\sin\alpha+\frac{c_1}{\sin\alpha}(c_0+b_0)]\tan^{-1}\frac{y}{x}$$

$$+\frac{\varepsilon A_0}{2}[c_0(c_0b_1+c_1b_0) + (b_0c_1\cot\alpha+b_1\sin\alpha)\sin\alpha - b_1]$$

$$\cdot\log(x^2+y^2) - \frac{\varepsilon^2A_0{}^2\sin\alpha}{x^2+y^2}\{[\frac{x}{2}\log(x^2+y^2) + y\tan^{-1}\frac{y}{x}]$$

$$\cdot\sin\alpha + [x\tan^{-1}\frac{y}{x} - \frac{y}{2}\log(x^2+y^2)](c_0+b_0)\}$$

$$-\frac{\varepsilon^2A_0{}^2x}{x^2+y^2}[(M_0\sin\alpha-R_0c_0)\sin\alpha - (1+c_0b_0)c_0b_0$$

$$+ T_0\sin^2\alpha + b_0(b_0+c_0) - \sin^2\alpha\cdot\log\varepsilon A_0]$$

$$+\frac{\varepsilon^2A_0{}^2y}{x^2+y^2}[M_0c_0+R_0\sin\alpha - c_0T_0 - (1+c_0b_0)b_0$$

$$-(b_0+c_0)\log\varepsilon A_0]\sin\alpha \qquad (7.24)$$

This is the three-term outer expansion of the two-term inner expansion.

If equations (7.18), (7.20) and (7.24) are used, the strength of singularities γ_2, μ_2, and λ_2 can be obtained.

VII. 5. Third-Order Free-Surface Expression

The equation (7.3) gives:

$$\eta_2 = \int_{-\infty}^{x} (\eta_1\phi_{1yy} + \phi_{2y} - \phi_{1x}\eta_{1x} - \phi_{1z}\eta_{1z})\,dx \quad \text{on } y=0 \qquad (7.25)$$

where the radiation condition gives the lower limit of the integral to be $-\infty$.

The equations (4.11) and (4.19) give (Appendix B)

$$\phi_1(x, \pm 0, z) = \pm\frac{\pi}{2}\,\gamma_1(z) \qquad \text{in the wake on } y=\pm 0 \tag{7.26a}$$

This function is well defined and can be differentiated with respect to x and z directly at y=0 to give:

$$\phi_{1x}(x, \pm 0, z) = 0 \qquad x \neq 0 \tag{7.26b}$$

$$\phi_{1z}(x, \pm 0, z) = \begin{cases} \pm\frac{\pi}{2}\gamma_1{}'(z) & \text{inside the wake} \\ 0 & \text{outside the wake} \end{cases} \tag{7.26c}$$

and from Appendix D, it gives

$$\phi_{1y}(x, \pm 0, z) = -\frac{1}{4\pi}\int_{-1}^{1} \frac{d\gamma_1(\zeta)}{d\zeta}\left[\frac{1}{z-\zeta} + \frac{\sqrt{x^2+(z-\zeta)^2}}{x(z-\zeta)}\right]d\zeta \tag{7.26d}$$

Substitute equation (7.26) into (7.25) to give:

$$\eta_2 = \int_{-\infty}^{x} (\eta_1\phi_{1yy} + \phi_{2y})\,dx \qquad \text{on } y=0,\ x<0 \tag{7.27a}$$

Only this upstream condition is needed. However, if equations (7.26a) - (7.26c) are used, one obtains:

$$\phi_{1yy} = -\phi_{1xx} - \phi_{1zz} = 0 \qquad \text{for } y=0,\ x<0 \tag{7.27b}$$

Finally, second-order free-surface deflection becomes:

$$\eta_2 = \int_{-\infty}^{x} \phi_{2y} \cdot dx \qquad \text{on } y=0,\ x<0 \qquad (7.27c)$$

So the three-term outer solution is:

$$\begin{aligned}\eta_0+\eta_1+\eta_2 \sim -\frac{1}{4\pi}\int_{-1}^{1} \frac{d\gamma_1(\zeta)}{d\zeta}\cdot\frac{1}{z-\zeta}\,[\ x+\sqrt{x^2+(z-\zeta)^2} \\ -|z-\zeta|\cdot\log\left|\frac{(z-\zeta)\ +\ \sqrt{x^2+(z-\zeta)^2}}{x}\right|]\,d\zeta \\ +\int_{-\infty}^{x} \phi_{2y}\cdot dx \qquad \text{on } y=0,\ x<0\end{aligned} \qquad (7.28)$$

Now, consider the three-term outer expansion of the two-term inner expansion for the free-surface deflection. Set $\zeta=\xi$ in equation (7.21a). Then separate the real and imaginary parts and eliminate ξ, the free-surface elevation for large ξ is thus obtained.

In order to perform the free-surface matching, equation (7.27c) must be evaluated near the line of singularities. Unfortunately, due to the mathematical difficulty, this inner limit has not yet been obtained. However, if this limit is evaluated, then its matching with the three-term outer expansion of the two-term inner expansion for free-surface will provide an equation to relate the depth of submergence and second-order parameters a_1, b_1, c_1, and thus give an unique second-order solution.

VIII. WEDGE SHAPE PLANING SURFACE

VIII. 1. Introducation

In the previous chapters, it is assumed that there is no deadrise angle and the planing surface takes the simplest form - a flat plate.

In the present chapter, the theory will be extended to include the case where the deadrise angle is different from zero. This will be the case for a V-shape planing surface. It is also shown that the theory can also be carried over to any kind of planing shape if there is no camber in the longitudinal direction.

VIII. 2. Formulation of the Problem

Let θ be the deadrise angle. The trailing edge is lying on a plane normal to the incident flow.

For an ideal irrotational flow, there exists a velocity potential $\phi(x, y, z)$ which satisfies the continuity condition:

$$\text{(L)} \qquad \nabla^2\phi = 0 \qquad \text{in fluid} \qquad (8.1)$$

The Bernoulli equation on the free surface:

$$\text{(F1)} \qquad \phi_x^{\,2} + \phi_y^{\,2} + \phi_z^{\,2} = 1 \qquad \text{on } S_f \qquad (8.2)$$

and the kinematic free-surface condition:

$$\phi_x \eta_x - \phi_y + \phi_z \eta_z = 0 \qquad \text{on } S_f \qquad (8.3)$$

Next consider the body condition. Define the wedge surface by the equation

$$y = S(x, z) \qquad (8.4a)$$

or $\quad \bar{S} = S(x, z) - y = x\tan\alpha + z\tan\theta - y = 0 \qquad (8.4b)$

Then the body condition is:

$$\frac{\partial\phi}{\partial n} = 0 = \phi_x \bar{S}_x + \phi_y \bar{S}_y + \phi_z \bar{S}_z \qquad (8.5)$$

Substitute (8.4b) into (8.5) to give:

$$\text{(B)} \qquad \phi_x \tan\alpha - \phi_y + \phi_z \tan\theta = 0 \qquad \text{on body} \qquad (8.6)$$

The radiation condition at upstream infinity gives:

$$\text{(R)} \qquad \begin{array}{l} \phi_x = 1; \\ \phi_y = 0; \\ \phi_z = 0 \end{array} \qquad \text{as } x \to -\infty, y \to -\infty \qquad (8.7)$$

The boundary-value problem for the wedge shape planing surface is thus mathematically formulated by (8.1), (8.2),

(8.3), (8.6), and (8.7).

Let the aspect-ratio AR be defined as

$$AR = \frac{(\text{Span Projected on the Undisturbed Free Surface})^2}{\text{Wetted Planing Area}} \tag{8.8}$$

VII. 3. Far-Field Boundary-Value Problem and Solution

Far away from the planing surface, the detail of the body shape is lost and the work done in the previous chapter for the planing plate is good for the wedge-shape planing surface as well with only one modification, namely, h is not constant any more but a function of z.

VIII. 4. Near-Field Boundary-Value Problem

The location of trailing edge is given by

$$y = h(z) = h_0 + z\tan\theta \tag{8.9}$$

where h_0 is the height of the trailing edge at the middle section above the undisturbed free surface.

Let the inner variables be

$$X = \frac{x+h(z)\cot\alpha}{\varepsilon}\ ;$$
$$Y = \frac{y-h(z)}{\varepsilon}\ ; \qquad Z = z \tag{8.10}$$

The inner boundary-value problem is:

$$\text{(L)} \qquad \Phi_{XX} + \Phi_{YY} + \varepsilon^2\Phi_{ZZ} = 0 \qquad \text{in fluid} \qquad (8.11)$$

$$\text{(F1)} \qquad \Phi_X{}^2 + \Phi_Y{}^2 + \varepsilon^2\Phi_Z{}^2 = 1 \qquad \text{on } S_f \qquad (8.12)$$

$$\text{(F2)} \qquad \frac{1}{\varepsilon}\Phi_X N_X - \Phi_Y + \varepsilon\Phi_Z N_Z = 0 \qquad \text{on } S_f \qquad (8.13)$$

$$\text{(B)} \qquad \Phi_X \tan\alpha - \Phi_Y + \varepsilon\Phi_Z \tan\theta = 0 \text{ on body} \qquad (8.14)$$

Assume the existence of asymptotic series for Φ and N as given in (2.22).

A. First-Order Inner Solution

The first-order boundary value problem is exactly the same as given in section III. 1. Therefore, the solution is given by (3.10) and (3.19). Now the quantity h is a function of z.

B. Second-Order Inner Solution

The boundary-value problem is

$$\text{(L)} \qquad \Phi_{1XX} + \Phi_{1YY} = 0 \qquad \text{in fluid} \qquad (8.15)$$

$$\text{(F1)} \qquad \Phi_{0X}\Phi_{1X} + \Phi_{0Y}\Phi_{1Y} = 0 \qquad \text{on } S_f \qquad (8.16)$$

$$\text{(F2)} \qquad \Phi_{0X}N_{1X} + \Phi_{1X}N_{0X} - \varepsilon\Phi_{1Y} = 0 \qquad \text{on } S_f \qquad (8.17)$$

$$\text{(B)} \qquad \Phi_{1X}\tan\alpha - \Phi_{1Y} + \varepsilon\Phi_{1Z}\tan\theta = 0 \quad \text{on body} \qquad (8.18)$$

It is noted that

$$h(z) = O(\ell(z))$$

so $$h(z) = O(\varepsilon) \qquad \text{as } \varepsilon\to 0 \tag{8.19}$$

The equations (8.9) and (8.19) give:

$$\tan\theta = o(1) \qquad \text{as } \varepsilon\to 0 \tag{8.20}$$

Use of (8.20) in (8.14) gives:

$$\Phi_{1X}\tan\alpha - \Phi_{1Y} = 0 \qquad \text{on body} \tag{8.21}$$

It is thus shown that the second-order inner boundary-value problem has exactly the same form as given in section IV. 3. The solution is thus given by (4.31) and (4.37).

VIII. 5. General-Shape Planing Surface with Curve in the Spanwise Direction

Previous sections show that the boundary-value problem for V-shape planing surface has exactly the same form as for the planing plate.

Now consider the general-shape planing surface given by the equation

$$y = x\tan\alpha + S(z) \tag{8.22}$$

Equation (8.22) implies that there is no camber in the longitudinal direction.

Next, it is to be shown that the theory up to the second-order solution presented in the previous chapter can be carried over directly to this general shape planing surface, with curve in the spanwise direction. In order to show this, it is sufficient to show that the body condition up to the second-order solution has the same form as given for the planing plate.

The body boundary condition gives:

$$\frac{\partial\phi}{\partial n} = 0 = \phi_x \tan\alpha - \phi_y + \phi_z S_z \qquad \text{on } y = x\tan\alpha + S(z) \tag{8.23}$$

Let N denote a unit vector lying in a cross-section plane, z= constant, the vector being normal to the contour of the section cut by the plane. Then the body boundary condition can be re-written

$$\frac{\partial\phi}{\partial N} = \frac{\tan\alpha\cdot\phi_x - \phi_y}{\sqrt{1+\tan^2\alpha}} \tag{8.24a}$$

or

$$\frac{\partial\phi}{\partial N} = -\frac{\phi_z\cdot S_z}{\sqrt{1+\tan^2\alpha}} \qquad \text{on } y = x\tan\alpha + S(z) \tag{8.24b}$$

In terms of inner variables, the left-hand side is $O(\frac{\phi}{\varepsilon})$, because of the differentiation; the right-hand side is $O(\varepsilon\phi)$ since S is $O(\varepsilon)$.

Let the solution have the inner expansion

$$\Phi(x,Y,Z) \sim \sum_{n=0}^{N} \Phi_n(X,Y;Z;\varepsilon)$$

where $\Phi_{n+1}=o(\Phi_n)$ as $\varepsilon\to 0$ for fixed X,Y,Z (8.25)

If (8.25) is substituted into (8.24b), the body condition is clearly,

$$\frac{\partial \Phi_0}{\partial N} = 0 \qquad \text{on body} \qquad (8.26)$$

To the second-order, it is noted that $\Phi_1=O(1)$. The left-hand side is $O(\frac{1}{\varepsilon})$ while right-hand side is $O(\varepsilon)$. Clearly,

$$\frac{\partial \Phi_1}{\partial N} = 0 \qquad \text{on body} \qquad (8.27)$$

Therefore, to the second-order inner solution, the body condition only depends on X and Y explicitly at a constant z-plane.

The theory developed for the planing plate can thus be applied directly to the general shape high-aspect-ratio planing surface with curve in the spanwise direction.

IX CONCLUSION

A high-aspect-ratio planing surface is investigated by the use of matched asymptotic expansions. A three dimensional effect is shown to provide a height reference for the planing surface so that the location of the planing surface is uniquely defined with respect to the undisturbed free surface. The numerical solution is presented in Chapter six.

In the present theory, it is assumed that the chord length distribution and its first derivative with respect to the span are continuous at the tip as elsewhere. The cusp shape distribution does satisfy those requirements. However, for practical applications it is desired to include a rectangular plate or elliptic-shape distribution. The last two shapes obviously violate the assumption of continuous first derivative at the tip. From analogies in aerodynamics, it is believed that the tip effect is only a local phenomenon, at least for the elliptic-shape chord distribution. Thus the theory presented here for elliptic-shape should provide a valid result through the whole span except in a very small neighborhood of the tips. Observation seems to confirm this statement even for a rectangular plate. If the tip effect were shown to be significant, a tip correction as suggested by Van Dyke[28]

might be applied.

In order to obtain a unique second-order solution, it is necessary to obtain a third-order outer solution; matching to the near-field and far-field free-surface elevations provides an equation relating the depth of submergence and the unknown second-order-problem parameters. Unfortunately, due to a mathematical difficulty, the inner limit of the third-order outer free-surface solution is not obtained in the present work. However, the method of solution is presented in Chapter seven.

The bottom of the planing surface is assumed to be flat through the first seven chapters. In Chapter eight, this restriction is relaxed to include a V-shape planing surface. To the first order, the present theory is applicable for arbitrary angle of deadrise. However, in the second-order solution, it is implicitly assumed that the deadrise angle is small.

Throughout the whole analysis, it is assumed that there is no camber in the longitudinal direction so that the classic hodograph method is applicable to obtain the inner solution. This restriction can also be relaxed in the manner as given by Wu.[15]

Gravity is neglected in the present theory by assuming a high speed planing surface with no wave behind it. In case the resistance of the planing surface is sought,

energy which is carried away by waves must be taken into account. To do this, Froude number must be considered. There will be two small parameters; one is the inverse of large aspect-ratio and the other is the inverse of large Froude number. To handle two parameters at the same time, the method developed by Ogilvie[33] may be applicable so that the gravity effect can be included.

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APPENDIX A

RELATIONSHIP BETWEEN Z-And ζ-PLANES

A. 1. To the First-Order Parameters

Equation (3.17) gives an expression between the physical Z-plane and the complex ζ-plane for any value of ζ. To perform the matching, only the outer limit is required for the inner solution. It is shown in section III. 3. that Z and ζ correspond to within a real scale factor far away from the planing surface. The right hand side of equation (3.17) can thus be expanded in terms of large ζ to give:

$$Z \sim A_0 e^{-i\alpha}\{(-c_0+\sqrt{1-c_0^2})\zeta + \log\zeta(1+c_0b_0-ib_0\sqrt{1-c_0^2})$$
$$-i\sqrt{1-c_0^2}(b_0\log 2 + \sqrt{b_0^2-1}\cdot\log(b_0-\sqrt{b_0^2-1})) - c_0$$
$$-(1+c_0b_0)\cdot\log(1+b_0)\} + O(\tfrac{1}{\zeta}) \tag{A-1}$$

Next, express ζ in terms of large Z.

$$\zeta \sim \frac{e^{i\alpha}}{A_0(-c_0+i\sqrt{1-c_0^2})} Z - \frac{1}{(-c_0+i\sqrt{1-c_0^2})}\{(1+c_0b_0-ib_0\sqrt{1-c_0^2})$$
$$\cdot\log\zeta - i\sqrt{1-c_0^2}[b_0\log 2 + \sqrt{b_0^2-1}\log(b_0-\sqrt{b_0^2-1})]$$
$$-c_0-(1+c_0b_0)\log(1+b_0)\} + O(\tfrac{1}{Z}) \tag{A-2}$$

where $A_0 = \dfrac{\delta_0}{\pi(b_0+c_0)}$

Now, take logarithm on both sides of equation (A-2) and

note that $\log|\zeta|=o(|\zeta|)$. Equation (A-2) becomes:

$$\zeta \sim \frac{e^{i\alpha}}{A_0(-c_0+i\sqrt{1-c_0^2})} Z + O(\log Z) \qquad \text{(A-3)}$$

If (A-3) is applied, first-order velocity potential matching gives:

$$c_0 = -\cos\alpha \qquad \text{(A-4)}$$

Substitute (A-4) into (A-1) to give:

$$Z \sim A_0\zeta + A_0(e^{-i\alpha} - b_0)\log\zeta$$
$$+ A_0 e^{-i\alpha}(M_0 - iR_0) + O(\tfrac{1}{\zeta}) \qquad \text{for large } \zeta \qquad \text{(A-5)}$$

where

$$M_0 = c_0 - (1+b_0c_0)\log(1+b_0)$$
$$R_0 = b_0\sqrt{1-c_0^2}\cdot\log 2 + \sqrt{b_0^2-1}\cdot\sqrt{1-c_0^2}\cdot\log(b_0-\sqrt{b_0^2-1}) \qquad \text{(A-6)}$$

Express ζ in terms of Z, one obtains the two-term outer limit of one-term inner solution for ζ in terms of Z:

$$\zeta = \frac{Z}{A_0} - (e^{-i\alpha}-b_0)\cdot\log\frac{Z}{A_0} - (M_0 - iR_0)e^{-i\alpha} + O(\tfrac{1}{Z}) \qquad \text{(A-7)}$$

where A_0 is real and positive.

A. 2. To the Second-Order Parameters

Equation (4.37) gives an expression between the Z-plane and the ζ-plane for any value of ζ. Following the same procedure given in Section A.1, after a long and tedious algebraic manipulation, one obtains the two-term outer limit of the two-term inner solution for Z in terms of ζ in equation (4.39a), and ζ in terms of Z in equation (4.39c); and the three-term outer limit of the two-term inner solution for Z in terms of ζ in equation (7.21), and ζ in terms of Z to three-term in equation (7.23).

APPENDIX B

To show that in the wake region, the density of doublets $\gamma(x,z)$ is a function of z only.

In the wake, the velocity potential ϕ is

$$\phi = \frac{1}{4\pi}\int_{-1}^{1} d\zeta \int_{0}^{\infty} d\xi \; \frac{y\cdot\gamma(\xi,\zeta)}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}} \qquad \text{(B-1)}$$

Let S represents the area of this integration. Next,consider this surface integral into two parts. Let

$$S = S^* + S' \qquad \text{(B-2)}$$

where S^* represents all the surface area in (B-1) except at a small circle of radius r with center at (x,0,z) and S' represents the area in this small circle.

$$\phi = \frac{1}{4\pi}\left[\iint_{S^*} + \iint_{S'}\right] \frac{y\cdot\gamma(\xi,\zeta)}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}} \qquad \text{(B-3)}$$

Consider the first integration S^*, since there is no singularities in the intergrand, this integral approaches zero as y approaches zero. Next, consider the integral in the small circle. When the radius r approaches zero, the area shrinks to a point. The density γ inside this small circle can be considered to be constant.

$$\phi \sim \frac{1}{4\pi} \lim_{r\to 0} \gamma(x,z) \iint_{S'} \frac{y \cdot d\xi \cdot d\zeta}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}} \tag{B-4}$$

The surface integral is shown to be zero if the area of small circle S' is excluded. Therefore, the surface of integration (B-4) can be extended to infinity to give:

$$\phi \sim \frac{1}{4\pi} \gamma(x,z) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y \; d\xi \; d\zeta}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}}$$

Integration gives:

$$\phi \sim \frac{1}{2} \gamma(x,z) \text{ sgn } y \qquad \text{as } y \to \pm 0 \tag{B-5}$$

Equation (4.2') gives

$$\phi_x(x,y,z) = 0 \qquad \text{on } y=0 \tag{B-6}$$

From equations (B-5) and (B-6), one obtains:

$$\gamma_x(x,z) = 0 \qquad \text{on } y=0$$

so

$$\gamma = \gamma(z) \qquad \text{on } y=0 \tag{B-7}$$

APPENDIX C

To obtain the inner limit of outer solution $\phi_0+\phi_1$.

The two-term outer solution is given in equation (4.11). In order to obtain the inner limit for this outer solution, this solution must be evaluated near the line of singularities and the equation becomes divergent integral. To avoid this difficulty, the method of integration by parts is used. Note that the strength of singularities is zero outside the planing body.

$$\gamma_1(z) = \mu_1(z) = \lambda_1(z) = 0 \qquad \text{at } z=+1,-1 \qquad \text{(C-1)}$$

Then equations (4.11) becomes:

$$\begin{aligned}\phi_0+\phi_1 \sim x &- \frac{1}{4\pi}\int_{-1}^{1} [\tan^{-1}\frac{y}{z-\zeta} + \tan^{-1}(\frac{y}{x}\cdot\frac{\sqrt{x^2+y^2+(z-\zeta)^2}}{z-\zeta})] \\ &\cdot\frac{d\gamma_1}{d\zeta}\cdot d\zeta + \frac{1}{4\pi}\cdot\frac{y}{x^2+y^2}\int_{-1}^{1}\frac{(z-\zeta)}{[x^2+y^2+(z-\zeta)^2]^{1/2}}\cdot\frac{d\mu_1}{d\zeta}d\zeta \\ &+ \frac{1}{4}\,\frac{x}{x^2+y^2}\int_{-1}^{1}\frac{(z-\zeta)}{[x^2+y^2(z-\zeta)^2]^{1/2}}\cdot\frac{d\lambda_1}{d\zeta}\,d\zeta \qquad \text{(C-2)}\end{aligned}$$

Take the inner variables in (C-2) to give:

$$\begin{aligned}\phi_0+\phi_1 \sim (\varepsilon X-\varepsilon H\cot\alpha) &- \frac{1}{4\pi}(\varepsilon Y+\varepsilon H)\int_{-1}^{1}\frac{d\gamma_1}{d\zeta}\cdot\frac{d\zeta}{z-\zeta} \\ &- \frac{1}{4\pi}\tan^{-1}\frac{\varepsilon Y+\varepsilon H}{(\varepsilon X-\varepsilon H\cot\alpha)}[\int_{1}^{z}\frac{d\gamma_1}{d\zeta}d\zeta - \int_{z}^{1}\frac{d\gamma_1}{d\zeta}\,d\zeta] \\ &+ \frac{1}{4\pi}\cdot\frac{\varepsilon Y+\varepsilon H}{(\varepsilon X-\varepsilon H\cot\alpha)^2+(\varepsilon Y+\varepsilon H)^2}\,2\mu_1(Z)\end{aligned}$$

$$+\frac{1}{4\pi}\cdot\frac{\varepsilon X-\varepsilon H\cot\ \alpha}{(\varepsilon X-\varepsilon H\cot\ \alpha)^2+(\varepsilon Y+\varepsilon H)^2}\cdot 2\lambda_1(Z)\ +o(\varepsilon\gamma_1)$$

If equation (4.10) is used, one obtains:

$$\begin{aligned}\phi_0+\phi_1 \sim\ &(\varepsilon X-\varepsilon H\cot\ \alpha)\ -\ \frac{1}{4\pi}(\varepsilon Y+\varepsilon H)\int_{-1}^{1}\frac{d\gamma_1}{d\zeta}\cdot\frac{d\zeta}{Z-\zeta}\\ &-\frac{\gamma_1(Z)}{2\pi}\cdot\tan^{-1}\left(\frac{\varepsilon Y+\varepsilon H}{\varepsilon X-\varepsilon H\cot\ \alpha}\right)\\ &+\frac{1}{2}\cdot\frac{(\varepsilon Y+\varepsilon H)\mu_1(Z)+(\varepsilon X-\varepsilon H\cot\ \alpha)\lambda_1(Z)}{(\varepsilon X-\varepsilon H\cot\ \alpha)^2+(\varepsilon Y+\varepsilon H)^2}\ +o(\varepsilon\lambda_1)\end{aligned} \quad \text{(C-3)}$$

APPENDIX D

To show that equation (5.16) gives:

$$\phi_1(x,0,z) = -\frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1(\zeta)}{d\zeta}\left[\frac{1}{z-\zeta} + \frac{\sqrt{x^2+(z-\zeta)^2}}{x(z-\zeta)}\right]\cdot d\zeta$$

Let $S^*=S+W$ represents the area of the planing surface and the wake. The second-order velocity potential is

$$\phi_1(x,y,z) = \frac{1}{4\pi}\iint_{S^*}\frac{y\gamma_1(\xi,\zeta)}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}}\,d\xi\cdot d\zeta \qquad \text{(D-1)}$$

Let

$$R = [(x-\xi)^2 + y^2 + (z-\zeta)^2]^{1/2} \qquad \text{(D-2)}$$

Then equation (D-1) becomes:

$$\phi_1(x,y,z) = -\frac{1}{4\pi}\iint_{S+W}\gamma_1(\xi,\zeta)\frac{\partial}{\partial y}\left(\frac{1}{R}\right)d\xi\cdot d\zeta$$

Next, differentiate ϕ_1 with respect to y.

$$\begin{aligned}\phi_{1y}(x,y,z) &= -\frac{1}{4\pi}\iint_{S+W}\gamma_1(\xi,\zeta)\frac{\partial^2}{\partial y^2}\left(\frac{1}{R}\right)d\xi\cdot d\zeta \\ &= \frac{1}{4\pi}\iint_{S+W}\gamma_1(\xi,\zeta)\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2}\right]\left(\frac{1}{R}\right)d\xi\cdot d\zeta\end{aligned} \qquad \text{(D-3)}$$

Next, apply the method of integration by parts in equation (D-3) to give:

$$\phi_{1y}(x,y,z) = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{\infty} \left[\frac{\partial\gamma_1}{\partial\xi}\cdot\frac{\partial}{\partial\xi}\left(\frac{1}{R}\right) - \frac{\partial\gamma_1}{\partial\zeta}\cdot\frac{\partial}{\partial\zeta}\left(\frac{1}{R}\right)\right]d\xi\cdot d\zeta$$

Next, change the order of integration and perform the integration by parts again to give:

$$\begin{aligned}\phi_{1y}(x,y,z) = &-\frac{1}{4\pi}\int_{L}^{\infty}\int_{-1}^{1} \frac{\partial^2\gamma_1}{\partial\xi\cdot\partial\zeta}\cdot\frac{(x-\xi)(z-\zeta)}{[(x-\xi)^2+y^2]R}\, d\xi\cdot d\zeta \\ &-\frac{1}{4\pi}\int_{-1}^{1}\frac{\partial\gamma_1}{\partial\zeta}\cdot\frac{z-\zeta}{y^2+(z-\zeta)^2}\cdot d\zeta \\ &-\frac{1}{4\pi}\int_{0}^{\infty}\int_{-1}^{1}\frac{\partial^2\gamma_1}{\partial\xi\partial\zeta}\cdot\frac{(x-\xi)(z-\zeta)}{[y^2+(z-\zeta)^2]R}\cdot d\xi\cdot d\zeta \qquad \text{(D-4)}\end{aligned}$$

In Appendix B, it shows that

$\gamma(x,z) = \gamma(z)$ in the wake region

$\gamma(x,z) = 0$ in front of the planing surface

Then, the strength of singularities can be represented by

$$\gamma(x,z) = \gamma(z)\cdot H(0)$$

where H(0) is the heaveside function.

$$\frac{\partial\gamma}{\partial x} = \gamma(z)\cdot\delta(0)$$

Therefore, $$\frac{\partial^2\gamma}{\partial\xi\cdot\partial\zeta} = \frac{\partial\gamma}{\partial\zeta}\cdot\delta(0) \qquad \text{(D-5)}$$

Substitute equation (D-5) into equation (D-4) to give:

$$\phi_{1y}(x,y,z) = -\frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1}{d\zeta}\{\int_{0}^{\infty}\delta(0)\,\frac{(x-\xi)(z-\zeta)}{[(x-\xi)^2+y^2]R}\cdot d\xi$$

$$+\frac{(z-\zeta)}{y^2+(z-\zeta)^2}+\int_{0}^{\infty}\delta(0)\frac{(x-\xi)(z-\zeta)}{[y^2+(z-\zeta)^2]R}\,d\xi\}d\zeta$$

$$= -\frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1}{d\zeta}\{\frac{x(z-\zeta)[x^2+2y^2+(z-\zeta)^2]}{(x^2+y^2)(y^2+(z-\zeta)^2\sqrt{x^2+y^2+(z-\zeta)^2})}$$

$$+\frac{z-\zeta}{y^2+(z-\zeta)^2}\}d\zeta \qquad \text{(D-6)}$$

Now, the integrand in equation (D-6) is well defined, so

$$\phi_{1y}(x,0,z) = -\frac{1}{4\pi}\int_{-1}^{1}\frac{d\gamma_1}{d\zeta}[\frac{\sqrt{x^2+(z-\zeta)^2}}{x(z-\zeta)}+\frac{1}{z-\zeta}]d\zeta \qquad \text{(D-7)}$$

where Cauchy Principal Value is taken.

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13. ABSTRACT A high-aspect-ratio planing surface gliding on a stream of an infinitely deep, incompressible, inviscid and gravity-free fluid is treated. This complicated problem is decomposed into two relatively simpler boundary-value problems: 1) The near-field boundary-value problem is valid only in the neighborhood of the planing surface. The problem is solved by the classical hodograph method. The second-order inner problem is also shown to be a plane, irrotational flow and the solution is obtained by following the same procedure as in the first-order inner solution. 2) The far-field boundary-value problem is valid only far away from the planing surface. The first-order outer solution is shown to be a trivial uniform flow. The outer velocity potential is defined in the whole space by harmonic continuation. The second-order solution is then shown to be similar to a lifting-line solution.

The unknown strength of singularities is obtained by matching of the velocity potential. Then a matching of the free-surface deflection provides a height reference for the planing surface. The location of the planing surface with respect to the undisturbed free surface is uniquely defined. In order to obtain a unique second-order solution, it is necessary to solve the third-order outer solution. The detail of this solution is presented.

A numerical solution for a planing plate of arbitrary angle of attack is presented. A downwash correction is also included. It is shown mathematically that the present theory can be applied to V-shape or general-shape planing surfaces with curvature in the spanwise direction.

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Planing surfaces						
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